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Some special configurations of points in $\mathbb{P}^n$

Anthony V. Geramita,[a,b] Tadahito Harima,[c] and Yong Su Shin[d,1]

[a] Department of Mathematics and Statistics, Queen's University, Kingston, ON, Canada K7L 3N6
[b] Dipartimento di Matematica, Universitá di Genova, Genova, Italy
[c] Department of Information Science, Shikoku University, Tokushima 771-11, Japan
[d] Department of Mathematics, Sungshin Women's University, Seoul, Republic of Korea 136-742

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1. Introduction

Let **H** be a numerical function which can be the Hilbert function of a finite set of points in $\mathbb{P}^n$. Following work of Macaulay, Hartshorne and others, one has a complete description of such functions (see, e.g., [16]).

However, for any particular **H** there are infinitely many different collections of points which share **H** as their Hilbert function. For reasons ranging from algebraic to geometric to combinatorial, there has always been a lively interest in determining what common features these different sets of points share as a consequence of sharing the same Hilbert function. One simply stated question of this type, that has been considered, is the following: If $\mathbb{X}$ is a set of points in $\mathbb{P}^n$ with Hilbert function **H**, does $\mathbb{X}$ have to have a certain number of points on a hypersurface of $\mathbb{P}^n$ of degree α (see [2])?

There has also always been interest in determining boundary conditions on collections of points sharing the same Hilbert function. Two simply stated questions of this type are the following: If $\mathbb{X}$ is a set of points in $\mathbb{P}^n$ with Hilbert function **H**, what is the maximal number of points of $\mathbb{X}$ which can lie on a hypersurface of $\mathbb{P}^n$ of degree α (see [12,14])?

If $\mathbb{X}$ is a set of points in $\mathbb{P}^n$ with Hilbert function **H**, what is the highest degree of a generator of the ideal of $\mathbb{X}$? (or more generally, how big can the graded Betti numbers of $\mathbb{X}$ be?) (see, e.g., [1,3,8,9,21–23,25,26].)

Another question which has also been considered is the following: Among all the sets of points in $\mathbb{P}^n$ with Hilbert function **H** is there at least one set $\mathbb{X}$ whose coordinate ring

E-mail addresses: tony@mast.queensu.ca, geramita@dima.unige.it (A.V. Geramita), harima@keiei.shikoku-u.ac.jp (T. Harima), ysshin@cc.sungshin.ac.kr, ysshin@mast.queensu.ca (Y.S. Shin).

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enjoys some particular algebraic property? For example, is at least one coordinate ring a Gorenstein ring or, more generally, a level ring (see [6,8,10–15,17,19,20,22,23,31])?

Two questions that have interested us a great deal are the following:

(A) If **H** is a possible Hilbert function for a set of reduced points in $\mathbb{P}^n$, what are all the possible graded Betti numbers for sets of points $\mathbb{X} \subset \mathbb{P}^n$ which have Hilbert function **H**?

(B) What can be the Hilbert function of a level Artinian algebra (see Section 2 for the definition of a level algebra)?

As for the first question: thanks to work of Bigatti [1], Hulett [21], and Pardue [24], we know that, given **H**, there is a unique maximal set of graded Betti numbers which $\mathbb{X}$ can have.

Our contribution to a solution to (A) is given in Section 2, where we define various kinds of "skew configurations" which one can associate to a given Hilbert function **H**. Roughly speaking, skew configurations build a point set having Hilbert function **H** from smaller sets of points having Hilbert functions derived from **H**. These smaller point sets are then put together carefully and this permits us to also know the minimal free resolution of the union from the same knowledge for the constituent pieces. We show, with various examples, how this works in practice.

Unfortunately, these "skew" constructions (as described for $\mathbb{P}^n$ in Section 2) do not give us much information directly on question (B). Thus, inasmuch as (B) is (essentially) a completely open question in codimension 3, we seek (in Section 3) a different approach.

In Section 3, we use the fact that if a set of points in $\mathbb{P}^2$ has level coordinate ring, then one can break the set up into two subsets and use those subsets to construct a level algebra of codimension 3—the first case of current interest. So, Section 3 begins by describing an analog of the "skew" constructions of Section 2, but which avoids the difficulties inherent in the methods of Section 2.

The more possibilities we can find for the Hilbert functions of subsets of the given level set of points, the more we can use the level set to make examples. Hence our interest in *special* level sets of points in $\mathbb{P}^2$ ("general" sets of points tend to have "general" subsets and, consequently, restricted possibilities for their Hilbert functions).

Thanks to a theorem of Iarrobino [22] we know all the possible Hilbert functions for level sets of points in $\mathbb{P}^2$. Thus, the key result of Section 3 is Theorem 3.12 which shows how to make special sets of points in $\mathbb{P}^2$ with level Hilbert function. We then use these methods to get a complete list (in codimension 3) of all Hilbert functions of level algebras of type 2 and socle degree $\leqslant 5$.

Our various constructions of special sets of points in $\mathbb{P}^n$ with given Hilbert function **H** lead us, in the final section, to consider the possibility that, for certain Hilbert functions **H**, *every* set of points with Hilbert function **H** is special.

After a small discussion (see Example 4.1) we show that this is the case for a very wide collection of Hilbert functions of points in $\mathbb{P}^2$. This is Theorem 4.7. It would be interesting to find similar such conditions in $\mathbb{P}^n$ $(n \geqslant 3)$.

2. Addition theorems for the Hilbert function and the resolution of a skew configuration

In [12], we defined special sets of points in $\mathbb{P}^n$ (called k-configurations) and showed how to use them, for example, to construct Gorenstein Artinian rings. In this section we would like to recall the notion of a k-configuration in $\mathbb{P}^n$ and show how their construction can be substantially generalized (to the notion of a skew configuration). This more general construction allows us (in general) to produce special sets of points with a given Hilbert function and different families of graded Betti numbers. The investigation of all the possibilities inherent in this construction seems a formidable task, which we do not undertake in this paper. Instead, we indicate a few ways this construction can be used to generate interesting examples. We also indicate some of the limitations inherent in this construction.

To understand the idea of a k-configuration, we begin as follows: let $\mathcal{S}_n$ be the collection of all the Hilbert functions of (reduced) point sets in $\mathbb{P}^n$. We have $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots$.

If $\mathbf{H} \in \mathcal{S}_n$ then $\Delta\mathbf{H}$ (the “first difference” of $\mathbf{H}$) is defined by:

$$\Delta\mathbf{H}(t) := \mathbf{H}(t) - \mathbf{H}(t-1).$$

Definition 2.1 [12]. If $\mathbf{H} \in \mathcal{S}_n$ then

$$\sigma(\mathbf{H}) := \text{the least integer } t \text{ for which } \Delta\mathbf{H}(t) = 0,$$

and

$$\alpha(\mathbf{H}) := \text{the least integer } t \text{ for which } \mathbf{H}(t) < \binom{t+n}{n}.$$

Remark 2.2. Since much of our focus is on the combinatorial properties present in a Hilbert function, there may be times when we simply refer to the Hilbert function as a succession of numbers which satisfy Macaulay’s growth condition (see [27,31]). As far as $\sigma(\mathbf{H})$ is concerned, there is no problem with this point of view since its definition is completely “internal” to the sequence $\mathbf{H}$. The number $\alpha(\mathbf{H})$, on the contrary, has an “external” reference to the number “n”, which is the dimension of the projective space in which we are considering the points which have Hilbert function $\mathbf{H}$. We thus remind the reader that $\alpha(\mathbf{H})$ depends on this prescription. The sequence

$$\mathbf{H}: 1\ 2\ 3\ 3\ \rightarrow$$

always satisfies $\sigma(\mathbf{H}) = 3$. If we consider $\mathbf{H}$ as the Hilbert function of 3 points in $\mathbb{P}^1$ we have $\alpha(\mathbf{H}) = 3$. If, alternatively, we consider $\mathbf{H}$ as the Hilbert function of 3 (necessarily collinear) points in $\mathbb{P}^n$ $(n \geqslant 2)$, then $\alpha(\mathbf{H}) = 1$.

In order not to encumber the notation too much we will always be very clear in specifying in which $\mathcal{S}_n$ we will consider $\mathbf{H}$, especially when we discuss $\alpha(\mathbf{H})$.

We now recall the definition of an *n-type vector.*

Definition 2.3 [12].

(1) A 0-*type vector* will be defined to be $\mathcal{T} = 1$. It is the only 0-*type vector.* We define $\alpha(\mathcal{T}) = -1$ and $\sigma(\mathcal{T}) = 1$.
(2) A 1-*type vector* is a vector of the form $\mathcal{T} = (d)$ where $d \geqslant 1$ is a positive integer. For such a vector we define $\alpha(\mathcal{T}) = d = \sigma(\mathcal{T})$.
(3) A 2-*type vector*, $\mathcal{T}$, is $\mathcal{T} = ((d_1), (d_2), \dots, (d_m))$, where $m \geqslant 1$, and the (d_i) are 1-*type vectors.* We also insist that $\sigma(d_i) < \alpha(d_{i+1})$. For such a $\mathcal{T}$ we define $\alpha(\mathcal{T}) = m$ and $\sigma(\mathcal{T}) = \sigma((d_m)) = d_m$. Clearly, $\alpha(\mathcal{T}) \leqslant \sigma(\mathcal{T})$ with equality if and only if $\mathcal{T} = ((1), (2), \dots, (m))$.

Remark. For simplicity in the notation we usually rewrite the 2-*type vector* $((d_1), \dots, (d_m))$ as $(d_1, \dots, d_m)$. In earlier papers, [12] and [17], we refered to this as the *alignment character*.

(4) Now let $n \geqslant 3$. An *n-type vector*, $\mathcal{T}$, is an ordered collection of $(n-1)$-*type vectors*, $\mathcal{T}_1, \dots, \mathcal{T}_s$, i.e., $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_s)$ for which $\sigma(\mathcal{T}_i) < \alpha(\mathcal{T}_{i+1})$ for $i = 1, \dots, s-1$. For such a $\mathcal{T}$ we define $\alpha(\mathcal{T}) = s$ and $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_s)$.

In [12], we proved that there is a 1–1 correspondence

$$\chi_n : \mathcal{S}_n \to \{n\text{-type vectors}\},$$

where if $\mathbf{H} \in \mathcal{S}_n$ then $\alpha(\mathbf{H}) = \alpha(\chi_n(\mathbf{H}))$ and $\sigma(\mathbf{H}) = \sigma(\chi_n(\mathbf{H}))$.

If

$$\rho_n : \{n\text{-type vectors}\} \to \mathcal{S}_n$$

denotes the inverse to χ_n, then we also showed in [11] that if $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_r)$ is an *n-type vector* and $\mathbf{H} = \rho_n(\mathcal{T})$ then setting $\widetilde{\mathcal{T}} = (\mathcal{T}_1, \dots, \mathcal{T}_s)$ and $\mathcal{T}' = (\mathcal{T}_{s+1}, \dots, \mathcal{T}_r)$ one has that $\widetilde{\mathcal{T}}$ and $\mathcal{T}'$ are also *n-type vectors* and if $\widetilde{\mathbf{H}} = \rho_n(\widetilde{\mathcal{T}})$ and $\mathbf{H}' = \rho_n(\mathcal{T}')$ then

$$\mathbf{H}(t) = \widetilde{\mathbf{H}}\big(t - (r-s)\big) + \mathbf{H}'(t). \tag{2.1}$$

Now, let $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_r)$ be an *n-type vector* and suppose that $\mathcal{T}$ corresponds to $\mathbf{H} \in \mathcal{S}_n$. We associate to $\mathcal{T}$ (or $\mathbf{H}$) certain sets of points in $\mathbb{P}^n$, called k-configurations in $\mathbb{P}^n$, which have Hilbert function $\mathbf{H}$. We do this inductively.

Definition 2.4 (*k-configuration in $\mathbb{P}^n$*, [12]).

$\mathcal{S}_0$: The only element in $\mathcal{S}_0$ is $\mathbf{H} := 1 \to$. It is the Hilbert function of $\mathbb{P}^0$, which is a single point. That is the only k-configuration in $\mathbb{P}^0$.
$\mathcal{S}_1$: Let $\mathbf{H} \in \mathcal{S}_1$. Then $\chi_1(\mathbf{H}) = \mathcal{T} = (e)$ where $e \geqslant 1$. We associate to $\mathbf{H}$ any set of e distinct points in $\mathbb{P}^1$. Clearly any set of e distinct points in $\mathbb{P}^1$ has Hilbert function $\mathbf{H}$. A set of e distinct points in $\mathbb{P}^1$ will be called a *k-configuration in $\mathbb{P}^1$ of type $\mathcal{T} = (e)$*.

$\mathcal{S}_2$: Let $\mathbf{H} \in \mathcal{S}_2$ and let $\mathcal{T} = ((e_1), \ldots, (e_r)) = \chi_2(\mathbf{H})$, where $\mathcal{T}_i = (e_i)$ is a 1-*type vector.* Choose r distinct $\mathbb{P}^1$'s in $\mathbb{P}^2$, i.e., lines in $\mathbb{P}^2$, and label them $\mathbb{L}_1, \ldots, \mathbb{L}_r$. By induction we choose, on $\mathbb{L}_i$, a *k-configuration* in $\mathbb{P}^1$, call it $\mathbb{X}_i$, of type $\mathcal{T}_i = (e_i)$—each *k-configuration* chosen so that no point of $\mathbb{L}_i$ contains any point of $\mathbb{X}_j$ for $j < i$.

The set $\mathbb{X} = \bigcup \mathbb{X}_i$ is called a *k-configuration in* $\mathbb{P}^2$ *of type* $\mathcal{T}$.

$\mathcal{S}_n$ $(n > 2)$: Now suppose that we have defined a *k-configuration* of *type* $\widetilde{\mathcal{T}} \in \mathbb{P}^{n-1}$, where $\widetilde{\mathcal{T}}$ is any $(n-1)$-*type vector* associated to $\mathbf{G} \in \mathcal{S}_{n-1}$.

Let $\mathbf{H} \in \mathcal{S}_n$ and suppose that $\chi_n(\mathbf{H}) = \mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_r)$ where the $\mathcal{T}_i$ are $(n-1)$-*type vectors*. Then $\rho_{n-1}(\mathcal{T}_i) = \mathbf{H}_i$ and $\mathbf{H}_i \in \mathcal{S}_{n-1}$.

Consider $\mathbb{H}_1, \ldots, \mathbb{H}_r$ distinct hyperplanes in $\mathbb{P}^n$ and let $\mathbb{X}_i$ be a *k-configuration* in $\mathbb{H}_i$ of type $\mathcal{T}_i$ such that $\mathbb{H}_i$ does not contain any point of $\mathbb{X}_j$ for any $j < i$.

The set $\mathbb{X} = \bigcup \mathbb{X}_i$ is called a *k-configuration in* $\mathbb{P}^n$ *of type* $\mathcal{T}$.

An important observation (see [12] and [13] for proofs) is the following: if $\mathcal{T}$ is an n-type vector and $\mathcal{T} \leftrightarrow \mathbf{H} \in \mathcal{S}_n$ then

(i) any k-configuration, $\mathbb{X}$, of type $\mathcal{T}$ has Hilbert function $\mathbf{H}$;
(ii) more generally, if $\mathbb{X}$ is any k-configuration of type $\mathcal{T}$, then the graded Betti numbers in the minimal free resolution of the ideal of $\mathbb{X}$ are completely determined only by $\mathcal{T}$.

Thus k-configurations are special sets of points which can be constructed for any permissible Hilbert function.

We would like to generalize the notion of a k-configuration in $\mathbb{P}^n$. Notice that for a k-configuration the inductive construction used linear hypersurfaces of $\mathbb{P}^n$. We generalize the idea so as to use hypersurfaces of varying degree.

Let $R = k[x_0, \ldots, x_n] = \bigoplus_{i \geqslant 0} R_i$, where R_i is the set of all homogeneous polynomials in R of degree i. For a closed reduced subscheme $\mathbb{V}$ in $\mathbb{P}^n$ and a finite set $\mathbb{X}$ of points in $\mathbb{V}$, we put:

$$\alpha(\mathbb{X}) := \min\bigl\{i \mid \mathbf{H}(\mathbb{X}, i) < \dim_k R_i\bigr\};$$
$$\sigma(\mathbb{X}) := \min\bigl\{i \mid \mathbf{H}(\mathbb{X}, i-1) = \mathbf{H}(\mathbb{X}, i)\bigr\} = \min\bigl\{i \mid \Delta\mathbf{H}(\mathbb{X}, i) = 0\bigr\};$$
$$\alpha_{\mathbb{V}}(\mathbb{X}) := \min\bigl\{i \mid \mathbf{H}(\mathbb{X}, i) < \mathbf{H}(\mathbb{V}, i)\bigr\}.$$

We now define our generalization of k-configurations.

Definition 2.5. A finite set of points $\mathbb{X}$ in $\mathbb{P}^n$ is a *skew configuration* in $\mathbb{P}^n$ if $\mathbb{X}$ satisfies the following conditions: there exist subsets $\mathbb{X}_1, \ldots, \mathbb{X}_u$ $(u \geqslant 2)$ of $\mathbb{X}$ and distinct reduced hypersurfaces $\mathbb{V}_1, \ldots, \mathbb{V}_u$ of degrees $d_1, \ldots, d_u$ respectively such that

(1) $\mathbb{X}$ is the union of the subsets $\mathbb{X}_1, \ldots, \mathbb{X}_u$;
(2) $\mathbb{X} \subset \mathbb{V}_i$ and $\alpha(\mathbb{X}_i) = d_i$ for each i, $1 \leqslant i \leqslant u$;
(3) $\mathbb{V}_i$ does not contain any points of $\mathbb{X}_j$ for any $j < i$;
(4) $\sigma(\mathbb{X}_i) + \alpha(\mathbb{X}_{i+i}) \leqslant \alpha_{\mathbb{V}_{i+1}}(\mathbb{X}_{i+1})$ for all $i < u$.

In this case $\mathbb{X}$ is called a *skew configuration of degree* $(d_1, \dots, d_u)$.

Remark 2.6. Notice that if $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ is a k-configuration of type $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_u)$ then, considering $\mathbb{X}_{i+1}$ in $\mathbb{H}_{i+1}$ we have $\alpha(\mathcal{T}_{i+1}) = \alpha_{\mathbb{H}_{i+1}}(\mathbb{X}_{i+1})$. Since $\alpha(\mathbb{X}_{i+1}) = 1$, we have

$$\sigma(\mathbb{X}_i) + \alpha(\mathbb{X}_{i+1}) \leqslant \alpha_{\mathbb{H}_{i+1}}(\mathbb{X}_{i+1}).$$

I.e., k-configuration in $\mathbb{P}^n$ are skew configurations in $\mathbb{P}^n$ of degree $(1, \dots, 1)$.

Note that, in general, the converse is not true. This is one reason why skew configurations are very interesting and useful.

Consider the following set of points in $\mathbb{P}^3$: Choose two hyperplanes $\mathbb{H}_1$ and $\mathbb{H}_2$ of $\mathbb{P}^3$ and let $\mathbb{X}_1 \subset \mathbb{H}_1$ be 6 points on an irreducible conic curve in $\mathbb{H}_1$ and let $\mathbb{X}_2$ be 15 points in $\mathbb{X}_2$ which do not lie on a quartic curve in that $\mathbb{H}_2$ (e.g., 14 points on a unique quartic and 1 point off the quartic). If we further assume that none of these 15 points are in $\mathbb{H}_1$, then the union of $\mathbb{X}_1$ and $\mathbb{X}_2$ is a skew configuration of degree $(1, 1)$. In this case, $\sigma(\mathbb{X}_1) = 4,\ \alpha(\mathbb{X}_2) = 1$, and $\alpha_{\mathbb{H}_2}(\mathbb{X}_2) = 5$. Clearly this is not a k-configuration in $\mathbb{P}^3$.

Proposition 2.7 (Addition Theorem for Hilbert functions). *Let $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ be a skew configuration in $\mathbb{P}^n$. Then*

$$\begin{aligned}
\mathbf{H}(\mathbb{X}, t) &= \mathbf{H}\left(\mathbb{X}_1, t - \sum_{j=2}^{u} \alpha(\mathbb{X}_j)\right) + \mathbf{H}\left(\mathbb{X}_2, t - \sum_{j=3}^{u} \alpha(\mathbb{X}_j)\right) + \cdots \\
&\quad + \mathbf{H}\big(\mathbb{X}_{u-1}, t - \alpha(\mathbb{X}_u)\big) + \mathbf{H}(\mathbb{X}_u, t) \\
&= \sum_{i=1}^{u} \mathbf{H}\left(\mathbb{X}_i, t - \sum_{j=i+1}^{u} \alpha(\mathbb{X}_j)\right)
\end{aligned} \tag{2.2}$$

for all $t \geqslant 0$, where $\mathbf{H}(\mathbb{X}_i, j) = 0$ for all $j < 0$.

Proof. We shall prove this by induction on u. If $u = 1$, we are done. Now suppose $u > 1$ and let $\mathbb{Y} = \bigcup_{i=1}^{u-1} \mathbb{X}_i$. Then $\mathbb{Y}$ is also a skew configuration in $\mathbb{P}^n$, and so, by induction on u, we have

$$\begin{aligned}
\mathbf{H}(\mathbb{Y}, t) &= \mathbf{H}\left(\mathbb{X}_1, t - \sum_{j=2}^{u-1} \alpha(\mathbb{X}_j)\right) + \mathbf{H}\left(\mathbb{X}_2, t - \sum_{j=3}^{u-1} \alpha(\mathbb{X}_j)\right) + \cdots \\
&\quad + \mathbf{H}\big(\mathbb{X}_{u-2}, t - \alpha(\mathbb{X}_{u-1})\big) + \mathbf{H}(\mathbb{X}_{u-1}, t) \\
&= \sum_{i=1}^{u-1} \mathbf{H}\left(\mathbb{X}_i, t - \sum_{j=i+1}^{u-1} \alpha(\mathbb{X}_j)\right).
\end{aligned} \tag{2.3}$$

By Proposition 3.5 in [14], we have

$$\mathbf{H}(\mathbb{X}, t) = \mathbf{H}(\mathbb{X}_u, t) + \mathbf{H}\big(\mathbb{Y}, t - \alpha(\mathbb{X}_u)\big). \tag{2.4}$$

Hence, from Eqs. (2.3) and (2.4), we obtain Eq. (2.2), as we wished. □

Corollary 2.8. *Let* $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ *be a skew configuration. Then*

$$\sigma(\mathbb{X}) = \sigma(\mathbb{X}_u).$$

Proof. We note that

$$\alpha(\mathbb{X}_i) \leqslant \alpha_{\mathbb{H}_i}(\mathbb{X}_i) \leqslant \sigma(\mathbb{X}_i)$$

for all i.

It follows from the definition of a skew configuration that

$$\sigma(\mathbb{X}_i) + \alpha(\mathbb{X}_{i+1}) \leqslant \alpha_{\mathbb{H}_{i+1}}(\mathbb{X}_{i+1}),$$

i.e.,

$$\sigma(\mathbb{X}_i) + \alpha(\mathbb{X}_{i+1}) \leqslant \sigma(\mathbb{X}_{i+1})$$

for all $i < u$, and hence, summing this for $i, i+1, \ldots, u-1$, we get

$$\sigma(\mathbb{X}_i) + \sum_{j=i+1}^{u} \alpha(\mathbb{X}_j) \leqslant \sigma(\mathbb{X}_u)$$

for all $i < u$. Thus, by Proposition 2.7, we have $\sigma(\mathbb{X}) = \sigma(\mathbb{X}_u)$. □

Example 2.9. Let $\mathcal{C}$ be an irreducible quadratic curve in $\mathbb{P}^2$ and $\mathbb{X}$ be a finite set of points on $\mathcal{C}$. One can easily check that

$$\mathbf{H}(\mathbb{X}, t) = \min\big\{\mathbf{H}(\mathcal{C}, t), |\mathbb{X}|\big\} = \min\big\{2t+1, |\mathbb{X}|\big\}.$$

Hence

$$\sigma(\mathbb{X}) = \max\big\{t \mid 2t \leqslant |\mathbb{X}|\big\} + 1 = \big[|\mathbb{X}|/2\big] + 1$$

and

$$\alpha_{\mathcal{C}}(\mathbb{X}) = \begin{cases} |\mathbb{X}|/2 & \text{if } |\mathbb{X}| \text{ is even,} \\ (|\mathbb{X}|+1)/2 & \text{if } |\mathbb{X}| \text{ is odd.} \end{cases}$$

Let $\mathcal{C}_1, \ldots, \mathcal{C}_u$ be distinct irreducible quadratic curves in $\mathbb{P}^2$, and let $\mathbb{X}_i$ $(1 \leqslant i \leqslant u)$ be finite sets of points on $\mathcal{C}_i$ such that $\mathcal{C}_i$ does not contain any point of $\mathbb{X}_j$ for any $j < i$ and $|\mathbb{X}_i| + 6 \leqslant |\mathbb{X}_{i+1}|$ for any $i < u$. Then $\mathbb{X} = \bigcup_{i=1}^{n} \mathbb{X}_i$ is a skew configuration in $\mathbb{P}^2$.

Remark 2.10. Let $\mathbf{H} \in \mathcal{S}_n$ be any Hilbert function and let $\alpha = \alpha(\mathbf{H})$. If we choose positive integers $d_1, \ldots, d_u$ such that $\alpha = \sum_{i=1}^{u} d_i$, then there is always a skew configuration of degree $(d_1, \ldots, d_u)$ whose Hilbert function is $\mathbf{H}$. Thus, in general, one can find in this way many special sets of points in $\mathbb{P}^n$ whose Hilbert function is exactly $\mathbf{H}$.

To see why this is so, let $\mathbf{H} \leftrightarrow \mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_\alpha)$ and let

$$\begin{aligned}
\mathcal{T}^1 &= (\mathcal{T}_1, \ldots, \mathcal{T}_{d_1}),\\
\mathcal{T}^2 &= (\mathcal{T}_{d_1+1}, \ldots, \mathcal{T}_{d_1+d_2}),\\
&\ \ \vdots\\
\mathcal{T}^u &= \bigl(\mathcal{T}_{(\sum_{i=1}^{u-1} d_i)+1}, \ldots, \mathcal{T}_\alpha\bigr).
\end{aligned}$$

Then each $\mathcal{T}^j$ is the type vector of an $\mathbf{H}_j \in \mathcal{S}_n$ and by Eq. (2.1), the $\mathbf{H}_j$ satisfy the addition formula in Proposition 2.7.

Let $\mathbb{X}_j$ be a set of points having Hilbert function $\mathbf{H}_j$ and $\mathbb{V}_j$ be a reduced hypersurface of degree d_j which contains $\mathbb{X}_j$. If we further insist that $\mathbb{V}_j$ does not contain any points in $\bigcup_{i<j} \mathbb{X}_i$, then $\bigcup_{i=1}^{u} \mathbb{X}_i$ is a skew configuration with Hilbert function $\mathbf{H}$.

Clearly, the variety of such examples depends on our ability to find $\mathbb{X}_j$'s and $\mathbb{V}_j$'s satisfying the condition above. (There always exists at least one example coming from a k-configuration.)

Example 2.11. Let $\mathbb{L}_1$ be a line in $\mathbb{P}^2$ and $\mathbb{X}_1$ a set of 3-points on $\mathbb{L}_1$. Let $\mathcal{C}_2$ be a reduced quadratic curve in $\mathbb{P}^2$ which does not contain the 3-points of $\mathbb{X}_1$. Let $\mathbb{X}_2$ be a set of 9-points on $\mathcal{C}_2$. Furthermore let $\mathcal{C}_3$ be a reduced quadratic curve in $\mathbb{P}^2$ which does not contain the 12-points of $\mathbb{X}_1 \cup \mathbb{X}_2$, and let $\mathbb{X}_3$ be a set of 17-points on $\mathcal{C}_3$. Let $\mathbb{L}_4$ be a line in $\mathbb{P}^2$ which does not contain the 29-points of $\mathbb{X}_1 \cup \mathbb{X}_2 \cup \mathbb{X}_3$, and let $\mathbb{X}_4$ be a set of 10-points on $\mathbb{L}_4$. Then $\mathbb{X} = \bigcup_{i=1}^{4} \mathbb{X}_i$ is a skew configuration in $\mathbb{P}^2$. Using Proposition 2.7, we can calculate the Hilbert function of $\mathbb{X}$. It is:

$$\mathbf{H}_{\mathbb{X}}: 1\ 3\ 6\ 10\ 15\ 21\ 27\ 33\ 36\ 39\ 39\ \rightarrow .$$

Let $I_{\mathbb{X}}$ be the ideal of a finite set $\mathbb{X}$ of points in $\mathbb{P}^n$ and $\nu(I_{\mathbb{X}})$ the number of minimal generators of $I_{\mathbb{X}}$. Let $d_1, \ldots, d_t$ be the degrees of the minimal generators of $I_{\mathbb{X}}$ and let $\Delta(I_{\mathbb{X}})$ denote the multiset $\{d_1, \ldots, d_t\}$. Also, for an integer d, we denote by $\Delta(I_{\mathbb{X}}) + d$ the multiset $\{d_1 + d, \ldots, d_t + d\}$. In the same way as in the proof of Theorem 2.7 in [13], we obtain the following theorem.

Proposition 2.12. *Let* $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ $(u \geqslant 2)$ *be a skew configuration in* $\mathbb{P}^n$ *and set* $\mathbb{Y} = \bigcup_{i=1}^{u-1} \mathbb{X}_i$. *Then*

$$\begin{aligned}
\nu(I_{\mathbb{X}}) &= \nu(I_{\mathbb{Y}}) + \nu(I_{\mathbb{X}_u}) - 1 \quad \textit{and}\\
\Delta(I_{\mathbb{X}}) &= \bigl\{\Delta(I_{\mathbb{Y}}) + \deg \mathbb{V}_u, \Delta\bigl(\bar{I}_{\mathbb{X}_u}\bigr)\bigr\}\\
&= \bigl\{\Delta(I_{\mathbb{Y}}) + \alpha(\mathbb{X}_u), \Delta\bigl(\bar{I}_{\mathbb{X}_u}\bigr)\bigr\},
\end{aligned}$$

where $\mathbb{V}_u$ is the hypersurface in $\mathbb{P}^n$ satisfying the conditions of Definition 2.5 *and*

$$\bar{I}_{\mathbb{X}_u} = I_{\mathbb{X}_u}/I_{\mathbb{V}_u} \subset R/I_{\mathbb{V}_u}.$$

Proof. The proof is very similar to that of Theorem 2.7 [13]. □

Remark 2.13. One of the important features of k-configurations comes from the fact that we could always describe the Betti numbers in their minimal free resolutions. That turns out also to be true for skew configurations and thus we have that skew configurations can be a source of different examples of special sets of points in $\mathbb{P}^n$ with the same Hilbert function but different graded Betti numbers. First we'll prove that the graded Betti numbers of a skew configuration depend completely on the graded Betti numbers of its components. Then we'll show how to use this in examples.

Theorem 2.14 (Addition Theorem for Resolutions). *Let $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ $(u \geqslant 2)$ be a skew configuration in $\mathbb{P}^n$ and put $\mathbb{Y} = \bigcup_{i=1}^{u-1} \mathbb{X}_i$. Let*

$$0 \to \mathcal{D}_n \to \mathcal{D}_{n-1} \to \cdots \to \mathcal{D}_j \to \cdots \to \mathcal{D}_1 \to R \to R/I_{\mathbb{Y}} \to 0$$

and

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_j \to \cdots \to \mathcal{E}_1 \to R \to R/I_{\mathbb{X}_u} \to 0$$

be the minimal free resolutions of $R/I_{\mathbb{Y}}$ and $R/I_{\mathbb{X}_u}$, respectively. Then the minimal free resolution of $R/I_{\mathbb{X}}$ is:

$$0 \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_j \to \cdots \to \mathcal{F}_1 \to R \to R/I_{\mathbb{X}} \to 0,$$

where

$$\mathcal{F}_1 = \mathcal{D}_1\bigl(-\alpha(\mathbb{X}_u)\bigr) \oplus \mathcal{E}_1/R\bigl(-\alpha(\mathbb{X}_u)\bigr) \quad \textit{and}$$
$$\mathcal{F}_j = \mathcal{D}_j\bigl(-\alpha(\mathbb{X}_u)\bigr) \oplus \mathcal{E}_j$$

for all $2 \leqslant j \leqslant n$.

Proof. The proof is similar to that of Theorem 3.6 [13]. □

Corollary 2.15. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two skew configurations, $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ and $\mathbb{Y} = \bigcup_{i=1}^{u} \mathbb{Y}_i$ where $\alpha(\mathbb{X}_i) = \alpha(\mathbb{Y}_i)$ for all i. Suppose $I_{\mathbb{X}_u}$ and $I_{\mathbb{Y}_u}$ have the same graded Betti numbers in their minimal free resolutions and the same is true for $I_{\mathbb{X}'}$, $\mathbb{X}' = \bigcup_{i=1}^{u-1} \mathbb{X}_i$, and $I_{\mathbb{Y}'}$, $\mathbb{Y}' = \bigcup_{i=1}^{u-1} \mathbb{Y}_i$. Then $\mathbb{X}$ and $\mathbb{Y}$ have the same graded Betti numbers in their minimal free resolutions.*

Example 2.16. In this example, we would like to show how, using two different skew configurations of the same degree (but with the same Hilbert function) one can get different resolutions. Thus skew configurations are an important tool in resolving problems concerning the possible resolutions consistent with a given Hilbert function (e.g., [3,23,25, 26]).

Consider the following skew configuration in $\mathbb{P}^3$ which is NOT a k-configuration. Let $\mathbb{X}_1$ be a complete intersection in $\mathbb{P}^2$ of type $(2,3)$ and let $\mathbb{V}_1$ be a reduced conic through the six points. Let $\mathbb{X}_2$ be a set of 16 general points in $\mathbb{P}^2$ such that $\mathbb{V}_1 \cap \mathbb{X}_2 = \emptyset$. Then $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2$ is a skew configuration in $\mathbb{P}^2$. Since the Hilbert functions of $\mathbb{X}_1$ and $\mathbb{X}_2$, respectively, are

$$\mathbf{H}_{\mathbb{X}_1}: 1\ 3\ 5\ 6\ 6\ 6\ \to,$$

$$\mathbf{H}_{\mathbb{X}_2}: 1\ 3\ 6\ 10\ 15\ 16\ \to,$$

we have that, by Proposition 2.7, the Hilbert function $\mathbf{H}_{\mathbb{X}}$ of $\mathbb{X}$ is:

$$\mathbf{H}_{\mathbb{X}}: 1\ 4\ 9\ 15\ 21\ 22\ 22\ \to.$$

Moreover, since the minimal free resolutions of $R/I_{\mathbb{X}_1}$ and $R/I_{\mathbb{X}_2}$ are

$$0 \to R(-6) \to \begin{array}{c} R(-3)\\ \oplus\\ R(-4)\\ \oplus\\ R(-5)\end{array} \to \begin{array}{c} R(-1)\\ \oplus\\ R(-2)\\ \oplus\\ R(-3)\end{array} \to R \to R/I_{\mathbb{X}_1} \to 0,$$

$$0 \to \begin{array}{c} R^3(-7)\\ \oplus\\ R(-8)\end{array} \to \begin{array}{c} R^8(-6)\\ \oplus\\ R(-7)\end{array} \to \begin{array}{c} R(-1)\\ \oplus\\ R^5(-5)\end{array} \to R \to R/I_{\mathbb{X}_2} \to 0,$$

respectively, we have that, by Theorem 2.14, the minimal free resolution of $R/I_{\mathbb{X}}$ is:

$$0 \to \begin{array}{c} R^4(-7)\\ \oplus\\ R(-8)\end{array} \to \begin{array}{c} R(-4)\\ \oplus\\ R(-5)\\ \oplus\\ R^9(-6)\\ \oplus\\ R(-7)\end{array} \to \begin{array}{c} R(-2)\\ \oplus\\ R(-3)\\ \oplus\\ R(-4)\\ \oplus\\ R^5(-5)\end{array} \to R \to R/I_{\mathbb{X}} \to 0.$$

On the other hand, the minimal free resolution of R/I, where I is the ideal of a k-configuration in $\mathbb{P}^3$ with Hilbert function $\mathbf{H}_{\mathbb{X}}$, is:

$$0 \to \begin{matrix} R(-6) \\ \oplus \\ R^5(-7) \\ \oplus \\ R(-8) \end{matrix} \to \begin{matrix} R(-4) \\ \oplus \\ R^2(-5) \\ \oplus \\ R^{11}(-6) \\ \oplus \\ R^2(-7) \end{matrix} \to \begin{matrix} R(-2) \\ \oplus \\ R(-3) \\ \oplus \\ R(-4) \\ \oplus \\ R^6(-5) \\ \oplus \\ R(-6) \end{matrix} \to R \to R/I \to 0.$$

Remark 2.17. Notice that all examples constructed using skew configurations $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ always have in their minimal free resolution

$$0 \to \mathcal{F}_n \to \cdots \to \mathcal{F}_1 \to R \to R/I_{\mathbb{X}} \to 0$$

where $\operatorname{rank} \mathcal{F}_n \geqslant u$.

Thus, even when the Hilbert function **H** could possibly support a Gorenstein set of points, it would be impossible to construct such a set directly using skew configurations.

The smallest rank one might hope for is when $\operatorname{rank} \mathcal{F}_n = u$. This happens if and only if each $\mathbb{X}_i$ is a Gorenstein set of points.

We now wish to investigate how we can use skew configurations with Gorenstein components to construct special sets of points in $\mathbb{P}^n$ whose resolution ends with $\operatorname{rank} \mathcal{F}_n = u$. One such a way is to choose, as the components, u complete intersections. We investigate that construction now.

Definition 2.18. A finite set $\mathbb{X}$ is a *skew complete intersection configuration* (*skew ci configuration* for short) in $\mathbb{P}^n$ if $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$, $u \geqslant 2$, is a skew configuration and if, in addition, all the subsets $\mathbb{X}_i$ are complete intersections in $\mathbb{P}^n$. In this case, we denote the number of complete intersection subsets of $\mathbb{X}$ by $r(\mathbb{X}) := u$. If $\mathbb{X}_i$ is the complete intersection of hypersurfaces of degrees $\alpha_{i1} \leqslant \cdots \leqslant \alpha_{in}$, we write "$\mathbb{X}_i$ is a $\mathrm{CI}(\alpha_{i1}, \ldots, \alpha_{in})$."

As a corollary of Theorem 2.14, we obtain the following.

Corollary 2.19. *Let $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ be a skew ci configuration in $\mathbb{P}^n$ and let*

$$0 \to \mathcal{K}_{in} \to \mathcal{K}_{i,n-1} \to \cdots \to \mathcal{K}_{i1} \to R \to R/I_{\mathbb{X}_i} \to 0$$

be the Koszul resolution of $\mathbb{X}_i$ $(1 \leqslant i \leqslant u)$. Then the minimal free resolution of $\mathbb{X}$ is:

$$0 \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_1 \to R \to R/I_{\mathbb{X}} \to 0,$$

where

$$\begin{aligned}\mathcal{F}_1 = \mathcal{K}_{11}\bigl(-\bigl(\alpha(\mathbb{X}_2)+\cdots+\alpha(\mathbb{X}_u)\bigr)\bigr)\\ \oplus\bigl[\mathcal{K}_{21}/R\bigl(-\alpha(\mathbb{X}_2)\bigr)\bigr]\bigl(-\bigl(\alpha(\mathbb{X}_3)+\cdots+\alpha(\mathbb{X}_u)\bigr)\bigr)\\ \oplus\bigl[\mathcal{K}_{31}/R\bigl(-\alpha(\mathbb{X}_3)\bigr)\bigr]\bigl(-\bigl(\alpha(\mathbb{X}_4)+\cdots+\alpha(\mathbb{X}_u)\bigr)\bigr)\oplus\cdots\\ \oplus\bigl[\mathcal{K}_{u-1,1}/R\bigl(-\alpha(\mathbb{X}_{u-1})\bigr)\bigr]\bigl(-\alpha(\mathbb{X}_u)\bigr)\oplus\mathcal{K}_{u1}/R\bigl(-\alpha(\mathbb{X}_u)\bigr)\end{aligned}$$

and, for each i $(2\leqslant i\leqslant n)$,

$$\begin{aligned}\mathcal{F}_i = \mathcal{K}_{1i}\bigl(-\bigl(\alpha(\mathbb{X}_2)+\cdots+\alpha(\mathbb{X}_u)\bigr)\bigr)\\ \oplus\mathcal{K}_{2i}\bigl(-\bigl(\alpha(\mathbb{X}_3)+\cdots+\alpha(\mathbb{X}_u)\bigr)\bigr)\oplus\cdots\\ \oplus\mathcal{K}_{u-1,i}\bigl(-\alpha(\mathbb{X}_u)\bigr)\oplus\mathcal{K}_{ui}.\end{aligned}$$

Remark 2.20. From Corollary 2.19, one can easily check that

$$\nu(I_{\mathbb{X}}) = \operatorname{rank}\mathcal{F}_1 = (n-1)u+1 \quad\text{and}\quad \operatorname{rank}\mathcal{F}_i = u\times\binom{n}{i}$$

for all $2\leqslant i\leqslant n$. In particular $r(\mathbb{X}) = u$ is the Cohen–Macaulay type of $\mathbb{X}$. Of particular interest to us is the special case when $\operatorname{rank}\mathcal{F}_n = u$ and $\mathcal{F}_n = R^u(-t)$ for some t.

Definition–Proposition 2.21. Let $R = k[x_0,\dots,x_n]$ and let $A = R/I$ be a Cohen–Macaulay ring of dimension d. Let

$$0\to\mathcal{F}_{n-(d-1)}\to\cdots\to\mathcal{F}_1\to I\to 0$$

be a minimal free resolution of I. Then

(a) If $B = B_0\oplus\cdots\oplus B_\ell$ $(B_\ell\neq 0)$ is an Artinian algebra, then B is *level* if and only if $B_\ell = \operatorname{Ann}(B_1)$ (see [6,9] for a discussion).
(b) A is a *level algebra* if $\mathcal{F}_{n-(d-1)} = R^m(-s)$ for some $s>0$; $\operatorname{rank}\mathcal{F}_{n-(d-1)} =$ Cohen–Macaulay type of A.
(c) (i) If $\mathbb{X}$ is a non-degenerate set of points in $\mathbb{P}^n$, $A = R/I_{\mathbb{X}}$ its coordinate ring, then we say that ℓ is the *socle degree* of $\mathbb{X}$ if ℓ is the socle degree of the Artinian algebra $B = A/\overline{L}A$, where $\overline{L}$ is any linear non-zero-divisor of A.
   (ii) $\mathbb{X}$ is called a *level set* of points if $A = R/I_{\mathbb{X}}$ is a level algebra. In this case, the socle degree of $\mathbb{X}$ is $\ell = \sigma(\mathbb{X})+n-1$.
(d) If $\overline{L}$ is a linear non-zero divisor in $A = R/I$, then A is level if and only if $A/\overline{L}A\simeq A/(L, I_{\mathbb{X}})$ is level.
(e) A 0-dimensional differentiable O-sequence (equivalently, an O-sequence whose first difference is the Hilbert function of an Artinian algebra, see, e.g., [16]) $b = \{b_i\}_{i\geqslant 0}$ with $b_1 = n+1$, is called *level* if there is a level set of points in $\mathbb{P}^n$ with Hilbert function b.

Unfortunately, as we will see in the next few propositions, the use of skew configurations to construct level sets of points, even in $\mathbb{P}^2$, is very limited.

Remark 2.22. Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^2$ of type $(d_1, \ldots, d_m)$. Then the minimal free resolution of $R/I_{\mathbb{X}}$ is

$$\begin{aligned} 0 &\to R\big(-(d_1+m)\big) \oplus \cdots \oplus R\big(-(d_i+m-i+1)\big) \oplus \cdots \oplus R\big(-(d_m+1)\big) \\ &\to R(-m) \oplus R\big(-(d_1+m-1)\big) \oplus \cdots \oplus R\big(-(d_i+m-i)\big) \oplus \cdots \oplus R(-d_m) \\ &\to R \to R/I_{\mathbb{X}} \to 0 \end{aligned}$$

by Theorem 2.6 in [17] and since $\sigma(\mathbb{X}) + 1 = d_m + 1$. Hence $\mathbb{X}$ is level if and only if

$$d_1 + m = \cdots = d_i + m - i + 1 = \cdots = d_m + 1,$$

that is, if and only if

$$d_{i+1} - d_i = 1$$

for every $i = 1, \ldots, m-1$.

For example, if $\mathbb{X}$ is a k-configuration in $\mathbb{P}^2$ of type $(4, 5, 6, 7, 8, 9)$, then the minimal free resolution of $R/I_{\mathbb{X}}$ is

$$0 \to R^6(-10) \to R(-6) \oplus R^6(-9) \to R/I_{\mathbb{X}} \to R \to 0.$$

Remark 2.23. Let $\mathbb{X}$ be a $\mathrm{CI}(\alpha, \beta)$ in $\mathbb{P}^2$, then the first difference function of $\mathbf{H}(\mathbb{X}, t)$ has the following form:

$$1\ 2\ \cdots\ \alpha-1\ \alpha\ \cdots\ \alpha\ \alpha-1\ \cdots\ 2\ 1\ 0\ \cdots,$$

where the value α is first reached when $t = \alpha - 1$, the value $\alpha - 1$ is reached for the second time when $t = \beta$, and the value 0 is reached first when $t = \alpha + \beta - 1$. In general, if F is a form of degree α in $R = k[x_0, \ldots, x_n]$ and $\mathbb{Z}$ is a $\mathrm{CI}(\alpha, \beta, \alpha_3, \ldots, \alpha_n)$ in $\mathbb{P}^n$, then

$$\Delta^{n-1}\mathbf{H}(\mathbb{Z}, t)\colon\ 1\ 2\ \cdots\ \alpha-1\ \underset{(\alpha-1)\mathrm{th}}{\alpha}\ \cdots\ \underset{(\beta-1)\mathrm{th}}{\alpha}\ b\ \cdots\ ,$$

where $b \leqslant \alpha - 1$. In particular, if $\mathbb{V}$ is a hypersurface of degree α in $\mathbb{P}^n$ containing $\mathbb{Z}$, then $\alpha_{\mathbb{V}}(\mathbb{Z}) = \beta$.

Now we would like to find a necessary and sufficient condition for a skew ci configuration in $\mathbb{P}^2$ to be level.

Corollary 2.24. *Let $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ be a skew ci configuration in $\mathbb{P}^2$, where $\mathbb{X}_i$ is a* $\mathrm{CI}(\alpha_{i1}, \alpha_{i2})$ *in $\mathbb{P}^2$ and let $\mathbb{V}_i$ be a curve of degree α_{i1} which contains $\mathbb{X}_i$ for every $i = 1, \ldots, u$. Then $\mathbb{X}$ is level if and only if*

$$\begin{cases} \alpha_{i1} = 1 & \text{for all } 2 \leqslant i, \\ \alpha_{i+1,2} = \alpha_{i1} + \alpha_{i2} & \text{for all } 1 \leqslant i < u. \end{cases}$$

Proof. If $\mathbb{X}$ is level, then it follows from Corollary 2.19 that

$$(\alpha_{i1} + \alpha_{i2}) + \alpha_{i+1,1} + \cdots + \alpha_{u1} = (\alpha_{i+1,1} + \alpha_{i+1,2}) + \alpha_{i+2,1} + \cdots + \alpha_{u1} \tag{2.5}$$

for all $1 \leqslant i < u$. Hence $\alpha_{i+1,2} = \alpha_{i1} + \alpha_{i2}$ for all $1 \leqslant i < u$. Also, since $\sigma(\mathbb{X}_i) + \alpha(\mathbb{X}_{i+1}) \leqslant \alpha_{\mathbb{V}_{i+1}}(\mathbb{X}_{i+1})$, we have

$$(\alpha_{i1} + \alpha_{i2} - 1) + \alpha_{i+1,1} \leqslant \alpha_{i+1,2} \tag{2.6}$$

for all $1 \leqslant i < u$ by Remark 2.23. Hence

$$\alpha_{i+1,1} \leqslant 1$$

for all $1 \leqslant i < u$. Therefore, since $\alpha_{i1} \geqslant 1$, we obtain

$$\alpha_{i1} = 1 \tag{2.7}$$

for all $i \geqslant 2$.

The converse also follows from Corollary 2.19. □

Remark 2.25. Thus, to form level skew ci configurations $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ in $\mathbb{P}^2$, we can essentially choose $\mathbb{X}_1$ to be any complete intersection $\mathrm{CI}(a, b)$, but then $\mathbb{X}_2, \ldots, \mathbb{X}_u$ must all be of type $(1, b_j)$, $2 \leqslant j \leqslant u$, and moreover

$$b_2 = a + b, \quad b_j = b_{j-1} + 1, \quad 3 \leqslant j \leqslant u.$$

For example, we can make a level skew ci configuration $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2 \cup \mathbb{X}_3$ from $\mathbb{X}_1$ a $\mathrm{CI}(2, 2)$, $\mathbb{X}_2$ a $\mathrm{CI}(1, 4)$, and $\mathbb{X}_3$ a $\mathrm{CI}(1, 5)$. The Cohen–Macaulay type of such an $\mathbb{X}$ is, from Remark 2.20, equal to 3 and the socle degree of $\mathbb{X}$ is, in view of Definition–Proposition 2.21(b) and Corollary 2.8, equal to 5. In fact, from Corollary 2.19 we get that the graded Betti numbers in a minimal free resolution of such an $\mathbb{X}$ are given by

$$0 \to R^3(-6) \to R^2(-4) \oplus R^2(-5) \to R \to R/I_{\mathbb{X}} \to 0.$$

Unfortunately, even this rather limited construction method for level sets of points in $\mathbb{P}^2$ does not extend to $\mathbb{P}^n$, $n > 2$, as the following Proposition 2.26 shows.

Proposition 2.26. *Let $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ be a non-degenerate skew ci configuration in $\mathbb{P}^n$ with $n \geqslant 3$. Then $\mathbb{X}$ is level if and only if $u = 1$.*

Proof. If $u = 1$, then $\mathbb{X}$ is a complete intersection in $\mathbb{P}^n$ and hence $\mathbb{X}$ is level.

Now assume $u \geqslant 2$. Let $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ be a skew ci configuration in $\mathbb{P}^n$ $(n \geqslant 3)$, where $\mathbb{X}_i$ is a $\mathrm{CI}(\alpha_{i1}, \ldots, \alpha_{in})$ in $\mathbb{P}^n$ and let $\mathbb{V}_i$ be a hypersurface of degree α_{i1} which contains $\mathbb{X}_i$ for every $i = 1, \ldots, u$. If $\mathbb{X}$ is level, then it follows from Corollary 2.19 that

$$\left(\sum_{j=1}^{n} \alpha_{ij}\right) + \alpha_{i+1,1} = \left(\sum_{j=1}^{n} \alpha_{i+1,j}\right) \tag{2.8}$$

for all $1 \leqslant i < u$. Also, since $\sigma(\mathbb{X}_i) + \alpha(\mathbb{X}_{i+1}) \leqslant \alpha_{\mathbb{V}_{i+1}}(\mathbb{X}_{i+1})$, we have

$$\left[\left(\sum_{j=1}^{n} \alpha_{ij}\right) - (n-1)\right] + \alpha_{i+1,1} \leqslant \alpha_{i+1,2} \tag{2.9}$$

for all $1 \leqslant i < u$ by Remark 2.23. Hence

$$\alpha_{i+1,1} + \alpha_{i+1,3} + \cdots + \alpha_{i+1,n} \leqslant n-1$$

for all $1 \leqslant i < u$. Therefore, since $\alpha_{in} \geqslant \cdots \geqslant \alpha_{i1} \geqslant 1$, we obtain

$$\alpha_{ij} = 1 \tag{2.10}$$

for all $i \geqslant 2$ and $j = 1, \ldots, n$. Furthermore it follows from (2.8) and (2.10) that

$$\alpha_{i+1,2} = \left(\sum_{j=1}^{n} \alpha_{ij}\right) - n + 2$$

for all $1 \leqslant i < u$, that is

$$\sum_{j=1}^{n} \alpha_{ij} = n-1$$

for such i since $\alpha_{i+1,2} = 1$. But, since $\alpha_{in} \geqslant \cdots \geqslant \alpha_{i1} \geqslant 1$, we obtain

$$n \leqslant \sum_{j=1}^{n} \alpha_{ij} = n-1.$$

This is a contradiction. □

The results of this section demonstrate that one can use skew configurations to construct special sets of points in $\mathbb{P}^2$ with the same Hilbert function but with lots of different collections of graded Betti numbers. However, skew configurations are not so useful for constructing level sets of points.

In the next section, we give a different method to construct level sets of points. We give such a construction only in the case of points in $\mathbb{P}^2$ because our main interest is to construct level Artinian algebras of codimension 3 (the least codimension in which it is unknown what can be the Hilbert function of a level algebra.)

3. Level sets of points in $\mathbb{P}^2$ and Artinian level algebras of codimension 3

In an earlier work [20], one of us was able to use special configurations of points in $\mathbb{P}^2$ to construct Artinian Gorenstein rings of codimension 3. In this section we begin an exploration of how to use special configurations of points in $\mathbb{P}^2$ to construct Artinian level rings of codimension 3.

Key ingredients in the work of [20] were the notions of a basic configuration and a pure configuration. We recall those notions here.

Definition 3.1 [20, Definition 3.1].

(a) A finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ is called a *basic configuration* of type (d, e) if there exist distinct elements b_j, c_j in k such that

$$I_{\mathbb{X}} = \left(\prod_{j=1}^{d} (x - b_j z), \prod_{j=1}^{e} (y - c_j z) \right).$$

We denote $\mathbb{X} := \mathbb{B}(d, e)$.

(b) A finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ is called a *pure configuration* if there exist finite basic configurations $\mathbb{B}(d_1, e_1), \dots, \mathbb{B}(d_m, e_m)$ where $e_1 > \cdots > e_m$, which satisfy the following three conditions:

(i) $\mathbb{B}(d_i, e_i) \cap \mathbb{B}(d_j, e_j) = \emptyset$ if $i \neq j$,

(ii) $\mathbb{X} = \mathbb{B}(d_1, e_1) \cup \cdots \cup \mathbb{B}(d_m, e_m)$,

(iii) $\varphi(\mathbb{B}(d_i, e_i)) \supset \varphi(\mathbb{B}(d_{i+1}, e_{i+1}))$ for all $1 \leqslant i \leqslant m-1$, where $\varphi : \mathbb{P}^2 \setminus \{(1, 0, 0)\} \to \mathbb{P}^1$ is the map defined by sending the point (x, y, z) to the point (y, z). In this case, we denote $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$.

Example 3.2. Let

$$\mathbb{X} = \left\{ \begin{array}{cccccccc}
\bullet & \bullet & & & & & & \\
\bullet & \bullet & & & & & & \\
\bullet & \bullet & & & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array} \right. \qquad \text{and} \qquad \mathbb{Y} = \left\{ \begin{array}{ccccccccc}
 & & & & & & \bullet & \bullet & \\
 & & & & & & \bullet & \bullet & \\
 & & & & & & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & & & & \\
 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array} \right.$$

be sets of points in $\mathbb{P}^2$. Then $\mathbb{X} = \mathbb{B}(2,8) \cup \mathbb{B}(3,5) \cup \mathbb{B}(3,2)$ is a pure configuration in $\mathbb{P}^2$, but $\mathbb{Y}$ is NOT a pure configuration in $\mathbb{P}^2$ since $\mathbb{Y}$ does not satisfy condition (b) (iii) of Definition 3.1.

Remark 3.3. It is useful to compare the idea of a pure configuration to the other configurations we have discussed: k-configurations and skew configurations.

Notice that pure configurations in $\mathbb{P}^2$, when all $d_j = 1$, are k-configurations. On the other hand, a k-configuration in $\mathbb{P}^2$ of type $(r_1, \ldots, r_s)$ need not be a pure configuration because of condition (b) (iii) of Definition 3.1.

Note that a pure configuration $\mathbb{X} \subseteq \mathbb{P}^2$ made up of basic configurations $\mathbb{X}_1 = \mathbb{B}(2,3)$ and $\mathbb{X}_2 = \mathbb{B}(2,4)$ is not a skew configuration of degree $(2,2)$. In fact,

$$\alpha(\mathbb{X}_1) = \alpha(\mathbb{X}_2) = 2 \quad \text{and} \quad \sigma(\mathbb{X}_1) = 4.$$

If $\mathbb{V}$ is a curve of degree 2 containing $\mathbb{X}_2$, then

$$\alpha_{\mathbb{V}}(\mathbb{X}_2) = 4,$$

and hence we have

$$\sigma(\mathbb{X}_1) + \alpha(\mathbb{X}_2) = 4 + 2 = 6 \nleqslant 4 = \alpha_{\mathbb{V}}(\mathbb{X}_2).$$

On the other hand, a skew ci configuration need not satisfy condition (b) (iii) of Definition 3.1. Thus, although there are similarities between these constructions, they are all different.

In [20], there is a formula for the Poincaré series of a pure configuration. We recall that result here.

Lemma 3.4 [20, Lemma 4.4]. *Let $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ be a pure configuration in $\mathbb{P}^2$. Then*

(1) $F(\mathbb{X}, \lambda) = \sum_{i=1}^{m} \lambda^{v_{i-1}} \frac{(1-\lambda^{d_i})(1-\lambda^{e_i})}{(1-\lambda)^3}$, *where $v_0 = 0$ and $v_i = d_1 + \cdots + d_i$.*
(2) $\sigma(\mathbb{X}) = \max\{e_i + v_i - 1 \mid 1 \leqslant i \leqslant m\}$.

Let $b = \{b_i\}_{i \geqslant 0}$ be a 0-dimensional differentiable O-sequence with $b_1 = 3$. Since $\{\Delta b_i\}$ is an O-sequence with $\Delta b_1 = 2$, it follows that

$$\Delta b_i = \begin{cases} i+1 & \text{for } 0 \leqslant i < \alpha, \\ \alpha & \text{for } \alpha \leqslant i < \theta, \end{cases}$$

where $\alpha := \min\{i \mid \Delta b_{i-1} \geqslant \Delta b_i\}$ and $\theta := \min\{i \mid \Delta b_{i-1} > \Delta b_i\}$. Furthermore, we have

$$\alpha > \Delta b_\theta \geqslant \cdots \geqslant \Delta b_{\sigma-1} > \Delta b_\sigma = 0,$$

where $\sigma := \min\{i \mid \Delta b_i = 0\}$. Hence we get

$$\Delta^2 b_i = \begin{cases} 1 & \text{for } 0 \leqslant i < \alpha, \\ 0 & \text{for } \alpha \leqslant i < \theta, \\ 0 \text{ or negative} & \text{for } \theta \leqslant i \leqslant \sigma. \end{cases} \tag{3.1}$$

Let

$$h(z) = 1 + \Delta b_1 z + \cdots + \Delta b_{\sigma-1} z^{\sigma-1} = 1 + 2z + \cdots + \tau z^{\sigma-1}$$

where $\tau = \Delta b_{\sigma-1}$. Form the new polynomial

$$q(z) = \sum_{i=0}^{\sigma+1} q_i z^i = h(z)(1-z)^2.$$

Proposition 3.5. *With notation as above, the following are equivalent.*

(a) $h(z)$*, of degree* $\sigma - 1$*, is the* h*-polynomial of a level Artinian quotient of* $k[x, y]$ *with Cohen–Macaulay type* (*necessarily*) $\tau = \Delta b_{\sigma-1}$.
(b) $q_i \leqslant 0$ *for* $1 \leqslant i \leqslant \sigma$.
(c) $\Delta^2 b_\theta \geqslant \Delta^2 b_{\theta+1} \geqslant \cdots \geqslant \Delta^2 b_\sigma$.
(d) *If* k *is a field of characteristic* 0*, then* $b = \{b_i\}$ *is a* 0*-dimensional differentiable level* O*-sequence.*

Proof. We first learned of the equivalence between (a) and (b) from G. Valla (unpublished). We won't include the proof here since we don't use this equivalence in what follows.

The equivalence between (a) and (c) is due to A. Iarrobino (see Theorem 4.6A of [22]). Since that equivalence is in print, we omit that proof as well.

(a) ⇔ (d) Let I be a homogeneous ideal of $k[x, y]$ such that $k[x, y]/I$ is a level Artinian ring with the h-polynomial $h(z)$. Then, since k is a field of characteristic zero, it follows from Theorem 10 in [28] that there exists a radical ideal $I^* \subset k[x, y, z]$, which lifts I. It follows that the finite set $\mathbb{X}$, of points in $\mathbb{P}^2$, defined by I^* is a level set of points with Hilbert function $b = \{b_i\}$. Hence $b = \{b_i\}$ is a 0-dimensional differentiable level O-sequence.

Conversely, let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^2$ with Hilbert function $b = \{b_i\}$ and $I_{\mathbb{X}} \subset k[x, y, z]$ the ideal defining $\mathbb{X}$. We may assume that z is not a zero divisor in the coordinate ring of $\mathbb{X}$. Then we have that $I := (I_{\mathbb{X}}, z)/(z) \subset k[x, y]$ is the desired ideal. This completes the proof of this theorem. □

From the addition formula for the Hilbert function of a k-configuration in $\mathbb{P}^2$ (see Proposition 2.7), we can obtain the following proposition.

Proposition 3.6. *Let* $b = \{b_i\}$ *be a* 0*-dimensional differentiable level* O*-sequence and let* $\mathcal{T} = (\delta_1, \ldots, \delta_\alpha)$ *be the* 2*-type vector associated to* b*. Then*

$$\delta_{i+1} - \delta_i \leqslant 2$$

for all $i = 1, \ldots, \alpha - 1$.

Remark 3.7. In general, the converse of Proposition 3.6 is not true. For example, let b be a 0-dimensional differentiable O-sequence with the type vector $\mathcal{T} = (1, 3, 4, 6)$. Then

$$\Delta^2 b\colon\ 1\ 1\ 1\ 1\ -1\ -2\ -1\ 0\ \rightarrow\ ,$$

which is not level.

The following proposition is immediate from Lemma 3.4 in [20].

Proposition 3.8. *Let $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ be a pure configuration in $\mathbb{P}^2$. Then a minimal free resolution of $\mathbb{X}$ is*:

$$0 \to \bigoplus_{i=1}^{m} R(-p_i) \to \bigoplus_{i=1}^{m+1} R(-q_i) \to R \to R/I_{\mathbb{X}} \to 0,$$

where

$$q_1 = e_1, \qquad q_i = d_1 + \cdots + d_{i-1} + e_i \quad (2 \leqslant i \leqslant m),$$
$$q_{m+1} = d_1 + \cdots + d_m, \qquad p_i = q_i + d_i \quad (1 \leqslant i \leqslant m).$$

Example 3.9. Let $\mathbb{X} = \mathbb{B}(2, 8) \cup \mathbb{B}(3, 5) \cup \mathbb{B}(2, 3)$ be a pure configuration in $\mathbb{P}^2$. Then, by Proposition 3.8, the minimal free resolution of $I_{\mathbb{X}}$ is

$$0 \to R^3(-10) \to R^2(-7) \oplus R^2(-8) \to I_{\mathbb{X}} \to 0.$$

Corollary 3.10. *Let $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ be a pure configuration in $\mathbb{P}^2$. Then $\mathbb{X}$ is level if and only if*

$$e_i - e_{i+1} = d_{i+1}$$

for all $1 \leqslant i \leqslant m - 1$.

If $b = \{b_i\}_{i \geqslant 0}$ is a 0-dimensional differentiable level O-sequence with $b_1 \leqslant 3$ and $\mathbb{X}$ is any level set of points in $\mathbb{P}^2$ with Hilbert function b, then the graded Betti numbers in a minimal free resolution of $I_{\mathbb{X}}$ are determined by b and are independent of $\mathbb{X}$. This implies that there is a unique "*degree matrix*" associated to b (see [4]). Choosing generic forms in $k[x, y, z]$ whose degrees are those given by this degree matrix and looking at the ideal of maximal minors, we will get an ideal whose zero set is a level set of points in $\mathbb{P}^2$ with Hilbert function b. Unfortunately, we cannot (using this procedure) find level sets of points having subsets with special Hilbert function. Since one of our goals is to use level sets of points in $\mathbb{P}^2$ to construct level Artinian codimension 3 graded k-algebras, this is a serious defect. Because of that, we now show how, given a level Hilbert function (with $b_1 \leqslant 3$), to find level pure configurations with that Hilbert function.

First, we need the following lemma.

Lemma 3.11. *Let $b = \{b_i\}$ be a* 0*-dimensional differentiable* O*-sequence with $b_1 = 3$ and*

$$\Delta^2 b_\theta \geqslant \Delta^2 b_{\theta+1} \geqslant \cdots \geqslant \Delta^2 b_\sigma.$$

Put

$$\begin{aligned}
&m := -\Delta^2 b_\sigma,\\
&d_i := \sigma - \min\bigl\{ j \mid \theta \leqslant j \leqslant \sigma,\ \bigl(\Delta^2 b_j\bigr) + i - 1 < 0\bigr\} + 1 \quad \textit{for all } 1 \leqslant i \leqslant m,\\
&e_1 = \theta \quad \textit{and} \quad e_i := e_{i-1} - d_i \quad \textit{for all } 2 \leqslant i \leqslant m.
\end{aligned}$$

Then $e_i > 0$ for every $i = 1, \ldots, m$.

Proof. It is obvious that $e_1 = \theta > 0$. Let $\alpha = \min\{i \mid b_i < \binom{i+2}{2}\}$. We first show that

$$\alpha = d_1 + \cdots + d_m.$$

By definition, $d_i = \sigma - j_i + 1$ where $\theta = j_1 \leqslant j_2 \leqslant \cdots \leqslant j_m \leqslant \sigma$. Let $\mathcal{B}$ be the multiset $\{\Delta^2 b_\theta, \ldots, \Delta^2 b_\sigma\}$. Then

$$\begin{aligned}
j_2 - j_1 &= \text{the number of } -1\text{'s in the multiset } \mathcal{B},\\
j_3 - j_2 &= \text{the number of } -2\text{'s in the multiset } \mathcal{B},\\
&\ \ \vdots\\
j_m - j_{m-1} &= \text{the number of } -(m-1)\text{'s in the multiset } \mathcal{B},\\
\sigma - j_m + 1 &= \text{the number of } -m\text{'s in the multiset } \mathcal{B}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\alpha &= (j_2 - j_1) + 2(j_3 - j_2) + \cdots + (m-1)(j_m - j_{m-1}) + m(\sigma - j_m + 1)\\
&= -(j_1 + \cdots + j_m) + m\sigma + m\\
&= (\sigma - j_1 + 1) + \cdots + (\sigma - j_m + 1)\\
&= d_1 + \cdots + d_m.
\end{aligned}$$

Hence, for $2 \leqslant i \leqslant m$, we have

$$\begin{aligned}
e_i &= e_{i-1} - d_i = \bigl(e_1 - (d_2 + \cdots + d_{i-1})\bigr) - d_i\\
&= e_1 - (d_2 + \cdots + d_i) = \theta - (d_2 + \cdots + d_i)\\
&\geqslant \alpha - (d_2 + \cdots + d_i) = (d_1 + \cdots + d_m) - (d_2 + \cdots + d_i)\\
&\geqslant d_1 > 0,
\end{aligned}$$

which completes the proof of this lemma. □

Theorem 3.12. *With the notation as in Lemma* 3.11, $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ *is a level set with Hilbert function b.*

Proof. From Corollary 3.10 and Lemma 3.11, $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ is a level set. So we need to show that $F(b,\lambda) = F(\mathbb{X},\lambda)$, where $F(b,\lambda) := \sum_{i \geqslant 0} b_i \lambda^i$. We use induction on m. Let $m = 1$, i.e., $\Delta^2 b_j = -1$ for all $\theta \leqslant j \leqslant \sigma$. Then we have

$$\begin{aligned} F(b,\lambda) &= \frac{1+\lambda+\cdots+\lambda^{\sigma-\theta}-\lambda^{\theta}-\lambda^{\theta+1}-\cdots-\lambda^{\sigma}}{(1-\lambda)^2} = \frac{(1-\lambda^{\sigma-\theta+1})(1-\lambda^{\theta})}{(1-\lambda)^3} \\ &= F\bigl(\mathbb{B}(\sigma-\theta+1,\theta),\lambda\bigr) = F\bigl(\mathbb{B}(d_1,e_1),\lambda\bigr). \end{aligned}$$

Now assume $m \geqslant 2$ and we consider the sequence $b' = \{b'_i\}$ such that

$$F\bigl(b',\lambda\bigr) = \frac{F(b,\lambda) - F(\mathbb{B}(d_1,e_1),\lambda)}{\lambda^{d_1}}. \tag{3.2}$$

Since

$$F\bigl(\mathbb{B}(d_1,e_1),\lambda\bigr) = \frac{(1-\lambda^{d_1})(1-\lambda^{e_1})}{(1-\lambda)^3} = \frac{1+\lambda+\cdots+\lambda^{d_1-1}-\lambda^{e_1}-\cdots-\lambda^{\sigma}}{(1-\lambda)^2}$$

and

$$\begin{aligned} F(b,\lambda) &= \frac{1+\lambda+\cdots+\lambda^{\alpha-1}+\Delta^2 b_\theta \lambda^\theta+\cdots+\Delta^2 b_\sigma \lambda^\sigma}{(1-\lambda)^2} \\ &= \frac{1+\lambda+\cdots+\lambda^{\alpha-1}+\Delta^2 b_{e_1} \lambda^{e_1}+\cdots+\Delta^2 b_\sigma \lambda^\sigma}{(1-\lambda)^2} \quad (\because \theta = e_1), \end{aligned}$$

we have

$$\begin{aligned} &F(b,\lambda) - F\bigl(\mathbb{B}(d_1,e_1),\lambda\bigr) \\ &\quad = \lambda^{d_1}\left[\frac{1+\lambda+\cdots+\lambda^{\alpha-1-d_1}+(\Delta^2 b_{e_1}+1)\lambda^{e_1-d_1}+\cdots+(\Delta^2 b_\sigma+1)\lambda^{\sigma-d_1}}{(1-\lambda)^2}\right] \\ &\quad = \lambda^{d_1}\left[\frac{1+\lambda+\cdots+\lambda^{\alpha-1-d_1}+(\Delta^2 b_{\theta}+1)\lambda^{\theta-d_1}+\cdots+(\Delta^2 b_\sigma+1)\lambda^{\sigma-d_1}}{(1-\lambda)^2}\right] \end{aligned}$$

and hence

$$\Delta^2 b'_i = \begin{cases} \Delta^2 b_{i+d_1} & \text{for } i < \theta - d_1, \\ \Delta^2 b_{i+d_1} + 1 & \text{for } i \geqslant \theta - d_1. \end{cases}$$

It is not hard to see that this description means that b' is a 0-dimensional differentiable O-sequence and $-\Delta^2 b'_{\sigma'} = -(\Delta^2 b_\sigma + 1) = m - 1$ where $\sigma' := \min\{i \mid \Delta b'_i = 0\} = \sigma - d_1$.

Let $\theta' = \min\{i \mid \Delta^2 b'_i < 0\}$. Then, we have

$$\Delta^2 b'\colon\ 1\ \cdots \underset{(\alpha-d_1-1)\text{th}}{1}\ 0\ \cdots\ 0\ \Delta^2 b'_{\theta'}\ \cdots \underset{(\sigma-\theta)\text{th}}{\Delta^2} b'_{\sigma'}\ 0\ \cdots,$$

and $\theta' \geqslant \theta - d_1$. Since

$$0 > \Delta^2 b'_\theta \geqslant \cdots \geqslant \Delta^2 b'_{\sigma'}$$

and $-\Delta^2 b'_{\sigma'} = m-1$, by induction on m, we have that

$$F\bigl(b', \lambda\bigr) = F\left(\bigcup_{i=1}^{m-1} \mathbb{B}\bigl(d'_i, e'_i\bigr), \lambda\right)$$

where

$$\begin{aligned}
&d'_i := \sigma' - \min\bigl\{j \mid \theta' \leqslant j \leqslant \sigma',\ \bigl(\Delta^2 b'_j\bigr) + i - 1 < 0\bigr\} + 1 \quad \text{for all } 1 \leqslant i \leqslant m-1,\\
&e'_1 = \theta' \quad \text{and} \quad e'_i := e'_{i-1} - d'_i \quad \text{for all } 2 \leqslant i \leqslant m-1.
\end{aligned}$$

Since $\theta' \geqslant \theta - d_1$ for $1 \leqslant i \leqslant m-1$, we have

$$\begin{aligned}
d'_i &= \sigma' - \min\bigl\{j \mid \theta' \leqslant j \leqslant \sigma',\ \bigl(\Delta^2 b'_j\bigr) + i - 1 < 0\bigr\} + 1,\\
&= (\sigma - d_1) - \min\bigl\{j \mid \theta - d_1 \leqslant j \leqslant \sigma - d_1,\ \bigl(\Delta^2 b_{j+d_1} + 1\bigr) + i - 1 < 0\bigr\} + 1\\
&= \sigma - \min\bigl\{j + d_1 \mid \theta \leqslant j + d_1 \leqslant \sigma,\ \Delta^2 b_{j+d_1} + (i+1) - 1 < 0\bigr\} + 1\\
&= \sigma - \min\bigl\{j \mid \theta \leqslant j \leqslant \sigma,\ \Delta^2 b_j + (i+1) - 1 < 0\bigr\} + 1 = d_{i+1},
\end{aligned}$$

and

$$\begin{aligned}
e'_1 &= \min\bigl\{i \mid \Delta^2 b'_i < 0\bigr\} = \min\bigl\{j \mid \theta - d_1 \leqslant j \leqslant \sigma - d_1,\ \Delta^2 b'_j < 0\bigr\}\\
&= \sigma' - d_2 + 1 = \sigma - d_1 - d_2 + 1 = \sigma - (\sigma - \theta + 1) - d_2 + 1\\
&= \theta - d_2 = e_1 - d_2 = e_2.
\end{aligned}$$

Moreover, inductively, we also have that, for $i = 1, \ldots, m-1$,

$$e'_i = e'_{i-1} - d'_i = e_i - d_{i+1} = e_{i+1}.$$

Hence we obtain that

$$F\bigl(b', \lambda\bigr) = F\left(\bigcup_{i=1}^{m-1} \mathbb{B}\bigl(d'_i, e'_i\bigr), \lambda\right) = F\left(\bigcup_{i=2}^{m} \mathbb{B}(d_i, e_i), \lambda\right).$$

Thus, from Eq. (3.2) and Lemma 3.4, we obtain that

$$\begin{aligned}F(b,\lambda) &= F\big(\mathbb{B}(d_1,e_1),\lambda\big)+\lambda^{d_1}F\big(b',\lambda\big)\\ &= F\big(\mathbb{B}(d_1,e_1),\lambda\big)+\lambda^{d_1}F\left(\bigcup_{i=2}^{m}\mathbb{B}(d_i,e_i),\lambda\right)=F(\mathbb{X},\lambda),\end{aligned}$$

which completes the proof of this theorem. $\square$

We now apply Theorem 3.12 to a specific example.

Example 3.13. Let $b=\{b_i\}$ be the 0-dimensional differentiable O-sequence

$$\begin{array}{rcccccccccccccc} & & & & & & & & & \alpha & & \theta & & & \sigma \\ b{:} & 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 44 & 52 & 59 & 64 & 67 & 67 \to \\ \Delta b{:} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 8 & 8 & 7 & 5 & 3 & 0 \\ \Delta^2 b{:} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & -2 & -2 & -3 \end{array}$$

Thus we see that (in the notion of Theorem 3.12) $\alpha=8$, $\theta=10$, $\sigma=13$ and

$$\Delta^2 b_{10}=-1\geqslant\Delta^2 b_{11}=-2\geqslant\Delta^2 b_{12}=-2\geqslant\Delta^2 b_{13}=-3.$$

Hence, from Theorem 3.12, we have that

$$F(b,\lambda)=F\big(\mathbb{B}(4,10)\cup\mathbb{B}(3,7)\cup\mathbb{B}(1,6),\lambda\big)$$

and the pure configuration $\mathbb{X}=\mathbb{B}(4,10)\cup\mathbb{B}(3,7)\cup\mathbb{B}(1,6)$ is a level set with Hilbert function b.

If $r(A)$ is the Cohen–Macaulay type of a Cohen–Macaulay standard graded k-algebra A, and if $A=\bigoplus_{i\geqslant 0}A_i$ is an Artinian level algebra, then $r(A)=\dim_k A_{\sigma(A)-1}$, where $\sigma(A)=\min\{i\mid A_i=0\}$. If $\mathbb{Z}=\bigcup_{i=1}^{m}\mathbb{B}(d_i,e_i)$ is a level pure configuration in $\mathbb{P}^2$, then $r(\mathbb{Z})=r(R/I_{\mathbb{Z}})=m$.

Lemma 3.14. *Let $\mathbb{Z}$ be a level set of points in $\mathbb{P}^n$ and $\mathbb{X}$ a subset of $\mathbb{Z}$. Set $\mathbb{Y}:=\mathbb{Z}\setminus\mathbb{X}$. Then $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ is an Artinian level graded k-algebra with $\sigma(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}))=\sigma(\mathbb{Z})-1$ and $r(R/(I_{\mathbb{X}}+Y_{\mathbb{Y}}))\leqslant r(\mathbb{Z})$.*

Proof. This argument is immediate from the mapping cone construction. $\square$

As an immediate corollary, we have:

Corollary 3.15. *Let $\mathbb{Z}=\bigcup_{i=1}^{m}\mathbb{B}(d_i,e_i)$ be a level pure configuration in $\mathbb{P}^2$ and $\mathbb{X}$ a subset of $\mathbb{Z}$. Set $\mathbb{Y}:=\mathbb{Z}\backslash\mathbb{X}$. Then $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ is an Artinian level graded k-algebra with $\sigma(R/(I_{\mathbb{X}}+Y_{\mathbb{Y}}))=d_1+e_1-2$ and $r(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}))\leqslant m$.*

This corollary naturally suggests the following question.

Question 3.16. What are the possible Hilbert functions of Artinian level codimension three algebras constructed by Corollary 3.15?

The question has a very nice answer when $m = 1$. In this case, $I_{\mathbb{X}} + I_{\mathbb{Y}}$ must be a Gorenstein ideal and $\mathbb{Z}$ must be a basic configuration. We know (thanks to [19]) that *all* possible Gorenstein Hilbert functions in codimension three can be found using Corollary 3.15.

One would like to know to what extent this is true for $m > 1$, e.g., if $m = 2$.

In this case, by Corollary 3.15, $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = 1$ or 2. If $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = 1$, then $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is Gorenstein and the Hilbert function $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is symmetric.

Question 3.17. What are the possible Hilbert functions of Artinian level algebras of codimension 3 with $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = 2$?

Example 3.18. We now consider Questions 3.16 and 3.17 for the cases $\sigma = 3, 4, 5$, and 6.

We begin by recalling a special case of a theorem of Fröberg and Laksov of [8] that if A is a level Artinian algebra of codimension 3 and type 2 with $\sigma - 1 = t$, then

$$\mathbf{H}(A, i) \leqslant \min\left\{\binom{2+i}{2}, 2\binom{t-i+2}{2}\right\}.$$

$\sigma = 3$: There is only one sequence possible, namely

$$1\ 3\ 2\ 0.$$

We obtain it as follows: let

$$\mathbb{U} = \begin{cases} \bullet\ \bullet \\ \bullet\ \bullet\ * \\ *\ *\ *. \end{cases}$$

We then apply Proposition 3.5 to see that $\mathbb{U}$ is a level set of points. We then apply Corollary 3.15 using the decomposition indicated by the different symbols we used to indicate the points of $\mathbb{U}$.

$\sigma = 4$: The only possibilities for the Hilbert functions of level Artinian algebras of codimension 3 and type 2 are:

$$1\ 3\ \alpha\ 2\ 0\ \rightarrow$$

where $2 \leqslant \alpha \leqslant 6$.

We apply the main idea of [15] to show that certain sequences cannot be the Hilbert function of a level Artinian algebra of codimension 3.

Claim.

(a) *There is no level Artinian algebra A having the Hilbert function*

$$\mathbf{H} = (1, 3, \underbrace{2, 2, \ldots, 2}_{s \text{ times}}) \quad or \quad (1, 3, c_2, \ldots, c_t, \underbrace{2, 2, \ldots, 2}_{s \text{ times}})$$

where $c_i \geqslant 3$ for every $i \geqslant 2$ and $s \geqslant 2$.

(b) *There is no level Artinian algebra A having the Hilbert function $\mathbf{H}_A = (1, 3, 3, 4)$. More generally, there are no level Artinian algebras A having Hilbert function of the form $\mathbf{H}_A = (1, 3, 3, 4, c_4, \ldots, c_t)$.*

Proof of Claim. (a) First of all, we consider the sequence

$$\mathbf{H}_A = (1, 3, \underbrace{2, 2, \ldots, 2}_{s \text{ times}})$$

with $s \geqslant 2$. Let $\mathbf{H}$ be a 0-dimensional differentiable O-sequence with $\Delta\mathbf{H} = \mathbf{H}_A$. Then, the 3-type vector of $\mathbf{H}$ is $\mathcal{T} = (1; \sigma - 1, \sigma)$, and hence, by Theorem 3.2 in [18], the minimal free resolution of the ideal of k-configuration $\mathbb{X}$ in $\mathbb{P}^3$ of type $\mathcal{T}$ is:

$$\begin{array}{l} 0 \to R(-4) \oplus R^2\big(-(\sigma+2)\big) \to R^4(-3) \oplus R^4\big(-(\sigma+1)\big) \\ \quad \to R^4(-2) \oplus R^2(-\sigma) \to R \to R/I_{\mathbb{X}} \to 0. \end{array}$$

Since $\sigma \geqslant 4$, that is, $\sigma + 1 \geqslant 5 > 4$, $R(-4)$ cannot be eliminated from the resolution, and hence $\mathbf{H}_A$ cannot be a level O-sequence.

Now suppose

$$\mathbf{H}_A = (1, 3, c_2, \ldots, c_t, \underbrace{2, 2, \ldots, 2}_{s \text{ times}})$$

with $c_i \geqslant 3$ for every $i \geqslant 2$ and $s \geqslant 2$. Let $\mathbf{H}$ be the 0-dimensional differentiable O-sequence such that $\Delta\mathbf{H} = \mathbf{H}_A$ and let $\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_\alpha)$ be the type vector of $\mathbf{H}$, where $\mathcal{T}_i$'s are all 2-type vectors. Notice that, since $c_3 \geqslant 3$, $\alpha(\mathcal{T}_\alpha) \geqslant 3$. The only thing we are interested in here is the last 2-type vector $\mathcal{T}_\alpha$ of $\mathcal{T}$. In fact, $\mathcal{T}_\alpha$ is of the form

$$\mathcal{T}_\alpha = (\ldots, \sigma - s - 2, \sigma - 1, \sigma)$$

where $\sigma = \sigma(\mathbf{H}_A)$. Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^3$ of type $\mathcal{T}$ and let

$$0 \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1 \to R \to R/I_{\mathbb{X}} \to 0$$

be the minimal free resolution of $R/I_{\mathbb{X}}$ where

$$\mathcal{F}_2 = R^{\alpha_{21}}(-\beta_{21}) \oplus R^{\alpha_{22}}(-\beta_{22}) \oplus R^{\alpha_{23}}(-\beta_{23}) \oplus \cdots,$$

$$\mathcal{F}_3 = R^{\alpha_{31}}(-\beta_{31}) \oplus R^{\alpha_{32}}(-\beta_{32}) \oplus R^{\alpha_{33}}(-\beta_{33}) \oplus \cdots$$

with $\beta_{21} > \beta_{22} > \beta_{23} > \cdots$ and $\beta_{31} > \beta_{32} > \beta_{33} > \cdots$. Then, by Theorem 3.2 in [18], we have that

$$\beta_{21} = \sigma + 1, \qquad \beta_{22} = \sigma - s + 1,$$
$$\beta_{31} = \sigma + 2, \qquad \beta_{32} = \sigma - s + 2.$$

So if $\mathbf{H}_A$ is a level O-sequence, then $\beta_{21} = \beta_{32}$, that is, $\sigma + 1 = \sigma - s + 2$, then $s = 1$, which is a contradiction since $s \geqslant 2$. It follows that $R^{\alpha_{32}}(-\beta_{32})$ cannot be deleted from $\mathcal{F}_3$, and so $\mathbf{H}_A$ cannot be a level O-sequence.

(b) We consider the sequence $\mathbf{H}_A = (1, 3, 3, 4)$. Let $\mathbf{H}$ be a 0-dimensional differentiable O-sequence with $\Delta\mathbf{H} = \mathbf{H}_A$. Then, the minimal free resolution of the ideal of a k-configuration $\mathbb{X}$ in $\mathbb{P}^3$ with Hilbert function $\mathbf{H}$ is:

$$\begin{aligned} 0 &\to R(-4) \oplus R^4(-6) \to R^3(-3) \oplus R^9(-5) \to R^3(-2) \oplus R^5(-4) \\ &\to R \to R/I_{\mathbb{X}} \to 0, \end{aligned}$$

and hence $\mathbf{H}_A = (1, 3, 3, 4)$ cannot be a level Artinian sequence.

Now consider the sequence $\mathbf{H}_A = (1, 3, 3, 4, c_4, \ldots, c_t)$ and let $\mathbf{H}$ be a 0-dimensional differentiable O-sequence with $\Delta\mathbf{H} = \mathbf{H}_A$. Since $\mathbf{H}_A(2) = 3$, we have that the type vector of $\mathbf{H}$ is $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2)$ where $\mathcal{T}_1 = (1)$ and $\mathcal{T}_2 = (d_1, \ldots, d_{\alpha(\mathcal{T}_2)})$. Moreover, since $\mathbf{H}_A(3) = 4$, we see that $\alpha(\mathcal{T}_2) \geqslant 4$. Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^3$ of type $\mathcal{T}$ and let

$$0 \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1 \to R \to R/I_{\mathbb{X}} \to 0$$

be the minimal free resolution of $R/I_{\mathbb{X}}$ where

$$\mathcal{F}_2 = R^{\alpha_{21}}(-\beta_{21}) \oplus R^{\alpha_{22}}(-\beta_{22}) \oplus R^{\alpha_{23}}(-\beta_{23}) \oplus \cdots,$$
$$\mathcal{F}_3 = R^{\alpha_{31}}(-\beta_{31}) \oplus R^{\alpha_{32}}(-\beta_{32}) \oplus R^{\alpha_{33}}(-\beta_{33}) \oplus \cdots$$

with $\beta_{21} < \beta_{22} < \beta_{23} < \cdots$ and $\beta_{31} < \beta_{32} < \beta_{33} < \cdots$. Then, by Theorem 3.2 in [18], we have that

$$\beta_{21} = 3, \qquad \beta_{22} = \alpha(\mathcal{T}_2) + 1,$$
$$\beta_{31} = 4, \qquad \beta_{32} \geqslant 6.$$

So if $\mathbf{H}_A$ is a level O-sequence then $\beta_{22} = \beta_{31}$, that is, $\alpha(\mathcal{T}_2) = 3$, which is a contradiction. It follows that $R^{\alpha_{31}}(-4) = R(-4)$ cannot be eliminated from $\mathcal{F}_3$, and so $\mathbf{H}_A$ cannot be a level O-sequence, as we wished. □

This leaves the sequences

$$1\ 3\ \alpha\ 2\ 0,$$

where $3 \leqslant \alpha \leqslant 6$. All of these can be obtained by using Proposition 3.5 and Corollary 3.15, as we did above.

$$\text{For } 1\ 3\ 3\ 2\ 0 \text{ use } \mathbb{Z} = \left\{\begin{array}{llll} \bullet & \bullet & & \\ \bullet & \bullet & & \\ * & \bullet & \bullet & \bullet \\ * & * & \bullet & \bullet \end{array}\right., \quad \text{for } 1\ 3\ 4\ 2\ 0 \text{ use } \mathbb{Z} = \left\{\begin{array}{llll} \bullet & \bullet & & \\ \bullet & \bullet & & \\ * & * & \bullet & \bullet \\ * & * & \bullet & \bullet \end{array}\right.,$$

$$\text{for } 1\ 3\ 5\ 2\ 0 \text{ use } \mathbb{Z} = \left\{\begin{array}{llll} \bullet & \bullet & & \\ \bullet & \bullet & & \\ * & * & \bullet & \bullet \\ * & * & * & \bullet \end{array}\right., \quad \text{for } 1\ 3\ 6\ 2\ 0 \text{ use } \mathbb{Z} = \left\{\begin{array}{llll} \bullet & \bullet & & \\ * & \bullet & & \\ * & \bullet & * & \bullet \\ * & * & * & \bullet \end{array}\right..$$

$\sigma = 5$: In this case, things are a bit more complicated.

We need, at first, to consider all sequences:

$$1\ 3\ \alpha\ \beta\ 2\ 0$$

which satisfy the Fröberg and Laksov condition: so $\alpha \leqslant 6$ and $\beta \leqslant 6$. They are:

$$\begin{array}{llll} 1,3,2,2,2, & 1,3,3,2,2, & 1,3,3,3,2, & 1,3,3,4,2, \\ 1,3,4,2,2, & 1,3,4,3,2, & 1,3,4,4,2, & 1,3,4,5,2, \\ 1,3,5,2,2, & 1,3,5,3,2, & 1,3,5,4,2, & 1,3,5,5,2, \\ 1,3,5,6,2, & 1,3,6,2,2, & 1,3,6,3,2, & 1,3,6,4,2, \\ 1,3,6,5,2, & 1,3,6,6,2. & & \end{array}$$

We also know $\beta = 2$ is not possible by the above claim (a) and hence the remaining cases are

$$\begin{array}{llll} 1,3,3,3,2, & 1,3,3,4,2, & 1,3,4,3,2, & 1,3,4,4,2, \\ 1,3,4,5,2, & 1,3,5,3,2, & 1,3,5,4,2, & 1,3,5,5,2, \\ 1,3,5,6,2, & 1,3,6,3,2, & 1,3,6,4,2, & 1,3,6,5,2, \\ 1,3,6,6,2. & & & \end{array}$$

Of these 13 sequences, 1 3 3 4 2 and 1 3 6 3 2 are impossible by results of [15] and 1 3 5 3 2 is impossible by work in [10]. (Here we have used the Type Vector Package of [5].)

The remaining 10 cases can be constructed as follows (using Proposition 3.5 and Corollary 3.15)

$$\text{For } 1\ 3\ 3\ 3\ 2\ 0 \text{ use } \mathbb{Z} = \left\{\begin{array}{ll} * & \\ \bullet & * \\ \bullet & * \\ \bullet & * \\ \bullet & \bullet \\ \bullet & * \end{array}\right., \quad \text{for } 1\ 3\ 4\ 3\ 2\ 0 \text{ use } \mathbb{Z} = \left\{\begin{array}{ll} * & \\ \bullet & * \\ \bullet & * \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & * \end{array}\right.,$$

$$\text{for } 1\,3\,4\,4\,2\,0 \text{ use } \mathbb{Z} = \left\{ \begin{array}{llll} \bullet & & & \\ \ast & \ast & & \\ \ast & \bullet & & \\ \ast & \bullet & & \\ \bullet & \bullet & & \\ \bullet & \bullet & & \end{array} \right., \quad \text{for } 1\,3\,4\,5\,2\,0 \text{ use } \mathbb{Z} = \left\{ \begin{array}{llll} \ast & & & \\ \ast & & & \\ \ast & \ast & \bullet & \\ \bullet & \ast & \bullet & \\ \bullet & \ast & \bullet & \\ \bullet & \ast & \bullet & \end{array} \right.,$$

$$\text{for } 1\,3\,5\,4\,2\,0 \text{ use } \mathbb{Z} = \left\{ \begin{array}{llll} \ast & & & \\ \ast & \ast & & \\ \ast & \ast & & \\ \bullet & \bullet & & \\ \bullet & \bullet & & \\ \bullet & \bullet & & \end{array} \right., \quad \text{for } 1\,3\,5\,5\,2\,0 \text{ use } \mathbb{Z} = \left\{ \begin{array}{llll} \ast & & & \\ \ast & & & \\ \ast & \bullet & \bullet & \\ \ast & \bullet & \bullet & \\ \bullet & \ast & \bullet & \\ \bullet & \ast & \bullet & \end{array} \right.,$$

$$\text{for } 1\,3\,5\,6\,2\,0 \text{ use } \mathbb{Z} = \left\{ \begin{array}{llll} \bullet & \bullet & & \\ \ast & \bullet & & \\ \ast & \ast & \bullet & \bullet \\ \ast & \ast & \bullet & \bullet \\ \ast & \bullet & \bullet & \bullet \end{array} \right., \quad \text{for } 1\,3\,6\,4\,2\,0 \text{ use } \mathbb{Z} = \left\{ \begin{array}{llll} \bullet & & & \\ \bullet & & & \\ \ast & \ast & \bullet & \\ \bullet & \bullet & \ast & \\ \bullet & \bullet & \ast & \\ \bullet & \ast & \ast & \end{array} \right.,$$

$$\text{for } 1\,3\,6\,5\,2\,0 \text{ use } \mathbb{Z} = \left\{ \begin{array}{llll} \bullet & & & \\ \ast & & & \\ \ast & \ast & \bullet & \\ \bullet & \bullet & \bullet & \\ \bullet & \ast & \ast & \\ \bullet & \ast & \ast & \end{array} \right., \quad \text{for } 1\,3\,6\,6\,2\,0 \text{ use } \mathbb{Z} = \left\{ \begin{array}{llll} \bullet & \bullet & & \\ \ast & \bullet & & \\ \ast & \ast & \bullet & \bullet \\ \ast & \ast & \ast & \bullet \\ \ast & \bullet & \bullet & \bullet \end{array} \right..$$

$\sigma = 6$: Now things become considerably more complicated.

We must, at first, consider all sequences

$$1\;3\;\alpha\;\beta\;\gamma\;2\;0,$$

where $3 \leqslant \alpha \leqslant 6$, $3 \leqslant \beta \leqslant 6$, and $3 \leqslant \gamma \leqslant 10$, which are 79 possible Artinian O-sequences. However, using the above claim (a), we can eliminate 23 cases where $\gamma = 2$. Applying the results of [15] we can also eliminate 20 cases more among all possible 79 Artinian O-sequences.

Hence the only sequences we really need consider are the ones in Tables 1 and 2.

The sequences in Table 1 are all the possible Hilbert functions of Artinian level algebras of socle degree 5 and type 2. They can all be constructed using the method of Corollary 3.15. (The constructions in this case all appear in [11].)

All the sequences in Table 2, however, are *not* possible as Hilbert functions of Artinian level algebras. (See [10] for the details on this case.)

Table 1

1	3	3	3	3	2	1	3	4	4	3	2	1	3	4	4	4	2	1	3	4	5	4	2
1	3	4	5	5	2	1	3	5	5	4	2	1	3	5	5	5	2	1	3	5	6	4	2
1	3	5	6	5	2	1	3	5	6	6	2	1	3	5	7	5	2	1	3	5	7	6	2
1	3	6	6	4	2	1	3	6	6	5	2	1	3	6	6	6	2	1	3	6	7	4	2
1	3	6	7	5	2	1	3	6	7	6	2	1	3	6	8	5	2	1	3	6	8	6	2
1	3	6	9	5	2	1	3	6	9	6	2	1	3	6	10	6	2						

Table 2

1	3	4	5	3	2	1	3	4	5	6	2	1	3	5	4	3	2	1	3	5	5	3	2
1	3	5	5	6	2	1	3	5	7	4	2	1	3	6	4	3	2	1	3	6	5	3	2
1	3	6	5	4	2	1	3	6	5	5	2	1	3	6	6	3	2	1	3	6	8	4	2
1	3	6	10	5	2																		

4. Decompositions of Points in $\mathbb{P}^2$ with prescribed Hilbert functions

Up to this point we have shown how, given the Hilbert function $\mathbf{H}$ of a set of points in $\mathbb{P}^n$, one can construct special sets of points in $\mathbb{P}^n$ with those Hilbert functions.

This leaves open the question of whether, for certain Hilbert functions, *all* sets of points with that Hilbert function are special.

Let's illustrate this idea with an example.

Example 4.1. Let $\mathbf{H} \in \mathcal{S}_2$, $\mathbf{H}: 1\ 3\ 5\ 6\ 7\ \to$. Then $\Delta\mathbf{H}: 1\ 2\ 2\ 1\ 1\ 0\ \to$ and by [7] any set of 7 points with this Hilbert function must consist of 5 points on one line and the remaining 2 on another. This is special! I.e., every set of points in $\mathbb{P}^2$ with this Hilbert function can be simply described.

In this section we describe a large set of Hilbert functions which exhibit similar behavior.

To set the notation for this section, let $\mathbf{H} \in \mathcal{S}_2$ and suppose the associated 2-type vector is $\mathcal{T} = (d_1, \dots, d_m)$.

Recall (see the beginning of Section 2), $\mathcal{T} = \chi_2(\mathbf{H})$ and $\mathbf{H} = \rho_2(\mathcal{T})$.

Proposition 4.2 (Proposition 3.6 in [12]). *For $\mathcal{T}$ as above. Then the following are equivalent.*

(1) $\rho_2(\mathcal{T})$ *is the Hilbert function of a complete intersection* $\mathrm{CI}(a, b)$.
(2) $d_{i+1} - d_i = 2$ *for all* $i = 1, \dots, m-1$ *and* $\sum_{i=1}^{m} d_i = ab$.

In this case, $a = m$, $b = d_1 + m - 1$ *and* $d_i = b + (2i - 1) - a$ *for all* $i = 1, 2, \dots, a$.

Definition 4.3. We say that a 2-type vector $\mathcal{T} = (d_1, \dots, d_m)$ is a *complete intersection type vector* (*ci type vector* for short) if $m = 1$ or $d_{i+1} - d_i = 2$ for all $i = 1, 2, \dots, m-1$.

Notation–Definition 4.4. Let $\mathcal{T}^1 = (d_{11}, \ldots, d_{1r_1}), \ldots, \mathcal{T}^\ell = (d_{\ell 1}, \ldots, d_{\ell r_\ell})$ be 2-type vectors with $d_{ir_i} < d_{i+1,1}$ for all $i = 1, \ldots, \ell - 1$. Then we denote the 2-type vector

$$\mathcal{T} = (d_{11}, \ldots, d_{1r_1}, \ldots, d_{\ell 1}, \ldots, d_{\ell r_\ell})$$

by the notation

$$\mathcal{T} = \bigcup_{i=1}^{\ell} \mathcal{T}^i,$$

and say that $\bigcup_{i=1}^{\ell} \mathcal{T}^i$ is a *decomposition* of $\mathcal{T}$.

Example 4.5. Let $\mathcal{T} = (3, 6, 8)$. Then all the decompositions of $\mathcal{T}$ are as follows:

$$(3, 6, 8),\ (3, 6) \cup (8),\ (3) \cup (6, 8),\ \text{and } (3) \cup (6) \cup (8).$$

Proposition 4.6. *Let $\mathcal{T} = (d_1, \ldots, d_m)$ be a 2-type vector and let $\mathcal{T} = \bigcup_{i=1}^{\ell} \mathcal{T}^i$ be a decomposition of $\mathcal{T}$ such that $\mathcal{T}^i = (d_{i1}, \ldots, d_{ir_i})$ is a ci type vector. Let $\mathbb{X}_i$ be a* $\mathrm{CI}(r_i, d_{i1} + r_i - 1)$, $i = 1, \ldots, \ell$, *where* $I_{\mathbb{X}_i} = (F_i, G_i)$ $(\deg F_i = r_i,\ \deg G_i = d_{i1} + r_i - 1)$. *Suppose* $[\bigcup_{j<i} \mathbb{X}_j] \cap V(F_i) = \emptyset$, $i = 2, \ldots, \ell$. *Then $\mathbb{X} = \bigcup_{i=1}^{\ell} \mathbb{X}_i$ is a skew ci configuration of* $\deg(r_1, \ldots, r_\ell)$ *with Hilbert function* $\rho_2(\mathcal{T})$.

Proof. First, we show that $\mathbb{X} = \bigcup_{i=1}^{\ell} \mathbb{X}_i$ is a skew ci configuration of degree $(r_1, r_2, \ldots, r_\ell)$. It is sufficient to show that $\sigma(\mathbb{X}_i) + \alpha(\mathbb{X}_{i+1}) \leqslant \alpha_{\mathbb{H}_{i+1}}(\mathbb{X}_{i+1})$ for all $i < \ell$, where $\mathbb{H}_i$ is a hypersurface of degree r_i containing $\mathbb{X}_i$ for every $i = 1, 2, \ldots, \ell$. Since $\mathbb{X}_i = \mathrm{CI}(r_i, d_{i1} + r_i - 1)$, we see

$$\alpha(\mathbb{X}_i) = r_i, \qquad \sigma(\mathbb{X}_i) = 2r_i + d_{i1} - 2, \qquad \alpha_{\mathbb{H}_i}(\mathbb{X}_i) = d_{i1} + r_i - 1.$$

Hence, noting that $d_{i1} + 2r_i - 1 \leqslant d_{i+1,1}$, we have

$$\sigma(\mathbb{X}_i) + \alpha(\mathbb{X}_{i+1}) = 2r_i + d_{i1} - 2 + r_{i+1} \leqslant d_{i+1,1} + r_{i+1} - 1 = \alpha_{\mathbb{H}_{i+1}}(\mathbb{X}_{i+1}).$$

Next, we show that $\mathbf{H}(\mathbb{X}, t) = \rho_2(\mathcal{T})(t)$ for all $t \geqslant 0$. For an integer $d > 0$, let $\tau(d)$ be the infinite sequence

$$1\ 2\ \cdots\ d\ d\ \rightarrow$$

and $\tau(d)(j) = 0$ if $j < 0$. From the definition of ρ_2 (in the proof of Theorem 2.6 in [12]), we obtain

$$\begin{aligned}
\rho_2(\mathcal{T})(t) &= \tau(d_m)(t) + \tau(d_{m-1})(t-1) + \cdots + \tau(d_1)\big(t - (m-1)\big) \\
&= \sum_{i=1}^{\ell} \Bigg[\tau(d_{i1}) \Bigg(t - (r_i - 1) - \sum_{j=i+1}^{\ell} r_j \Bigg)
\end{aligned}$$

$$+\tau(d_{i2})\left(t-(r_i-2)-\sum_{j=i+1}^{\ell} r_j\right)+\cdots+\tau(d_{ir_i})\left(t-\sum_{j=i+1}^{\ell} r_j\right)\Bigg]$$

$$=\sum_{i=1}^{\ell}\rho_2\big(\mathcal{T}^i\big)\left(t-\sum_{j=i+1}^{\ell} r_j\right).$$

Also, since $\mathcal{T}^i$ is a ci type vector, it follows from Proposition 4.2 that

$$\rho_2\big(\mathcal{T}^i\big)\left(t-\sum_{j=i+1}^{\ell} r_j\right)=\mathbf{H}\left(\mathbb{X}_i, t-\sum_{j=i+1}^{\ell} r_j\right).$$

Furthermore, we note that $\alpha(\mathbb{X}_j)=r_j$. Hence, from Proposition 2.7, we get

$$\mathbf{H}(\mathbb{X},t)=\sum_{i=1}^{\ell}\mathbf{H}\left(\mathbb{X}_i, t-\sum_{j=i+1}^{\ell}\alpha(\mathbb{X}_j)\right)=\sum_{i=1}^{\ell}\mathbf{H}\left(\mathbb{X}_i, t-\sum_{j=i+1}^{\ell} r_j\right)=\rho_2(\mathcal{T})(t)$$

for all $t \geqslant 0$. □

Our main theorem of this section says that if $\mathcal{T}=(d_1,\ldots,d_m)$ is a 2-type vector for which $d_{i+1}-d_i \geqslant 2$ for every $i=1,\ldots,m-1$, then *all* sets of points with Hilbert function $\mathbf{H}=\rho_2(\mathcal{T})$ are special.

Theorem 4.7. *Let $\mathcal{T}=(d_1,\ldots,d_m)$ be a 2-type vector with $d_{i+1}-d_i \geqslant 2$ for each $i=1,\ldots,m-1$ and let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^2$ with Hilbert function $\rho_2(\mathcal{T})$. Then $\mathbb{X}=\bigcup_{i=1}^{\ell}\mathbb{X}_i$ is a skew ci configuration such that*

(1) *$\mathcal{T}=\bigcup_{i=1}^{\ell}\mathcal{T}^i$ is a decomposition of $\mathcal{T}$ into ci type vectors $\mathcal{T}^i$,*
(2) *the Hilbert function of $\mathbb{X}_i$ is $\rho_2(\mathcal{T}^i)$.*

Proof. The given 2-type vector $\mathcal{T}$ has a decomposition

$$\mathcal{T}=\bigcup_{i=1}^{s}(d_{i1},\ldots,d_{ir_i})=\big(\widetilde{\mathcal{T}}^1,\ldots,\widetilde{\mathcal{T}}^s\big)$$

with $d_{i,j+1}-d_{i,j}=2$ for all i, j and $d_{i+1,1}-d_{ir_i}>2$ for all i. By a theorem of Davis in [7], if $\mathbb{X}$ is any set of points with Hilbert function $\rho_2(\mathcal{T})$, then $\mathbb{X}=\bigcup_{i=1}^{s}\widetilde{\mathbb{X}}_i$ is a skew configuration for which the Hilbert function of $\widetilde{\mathbb{X}}_i$ is $\rho_2(\widetilde{\mathcal{T}}^i)$.

Thus, it remains to prove the theorem only for $s=1$, i.e., we may assume $\mathcal{T}=(d_1,\ldots,d_m)$ with $d_{i+1}-d_i=2$ for all $i=1,\ldots,m-1$. We use induction on $m \geqslant 1$.

Let $m=1$. Then, since $\mathbb{X}$ is d_1-points on a line, our assertion is trivial.

Now suppose $m \geqslant 2$ and $I=I_{\mathbb{X}}=\bigoplus_{i\geqslant 0} I_i$. We set $\beta(\mathbb{X}) := \min\{t \mid \text{height } J_t \geqslant 2\}$, where J_t is the ideal generated by $\bigcup_{i=0}^{t} I_i$.

Claim 1. $d_1 + m - 1 \leqslant \beta(\mathbb{X}) \leqslant d_m = d_1 + 2m - 2$.

Proof of Claim 1. Since $\mathbb{X}$ has the Hilbert function of a $\mathrm{CI}(m, d_1 + m - 1)$, we have

$$\Delta\mathbf{H}(\mathbb{X}, i) = \begin{cases} i + 1, & 0 \leqslant i < m - 1, \\ m, & m - 1 \leqslant i < d_1 + m - 1, \\ d_1 + 2m - 2 - i, & d_1 + m - 1 \leqslant i < d_m = d_1 + 2m - 2, \\ 0, & i \geqslant d_m = d_1 + 2m - 2. \end{cases} \tag{4.1}$$

Hence, by Proposition 3 in [29], we have

$$d_1 + m - 1 \leqslant \beta(\mathbb{X}) \leqslant d_m = d_1 + 2m - 2.$$

Set

$$D := \text{g.c.d}\bigl\{F \in I_{\mathbb{X}} \mid F \text{ is a form of degree } < \beta(\mathbb{X})\bigr\} \quad \text{and}$$
$$\mathbb{Y} := \bigl\{P \in \mathbb{X} \mid D(P) = 0\bigr\}. \quad \square$$

Claim 2. $d := \deg(D) = d_1 + 2m - 1 - \beta(\mathbb{X})$.

Proof of Claim 2. Since

$$\Delta\mathbf{H}\bigl(R/(D), t\bigr) = \Delta\mathbf{H}\bigl(R/(D, I_{\mathbb{X}}), t\bigr)$$

for every $t = 0, \dots, \beta(\mathbb{X}) - 1$, we have that

$$d = \Delta\mathbf{H}\bigl(R/(D), \beta(\mathbb{X}) - 1\bigr) = \Delta\mathbf{H}\bigl(R/(D, I_{\mathbb{X}}), \beta(\mathbb{X}) - 1\bigr).$$

Hence, by Proposition 14 in [30] and Eq. (4.1) above, we get

$$d = \Delta\mathbf{H}\bigl(R/(D, I_{\mathbb{X}}), \beta(\mathbb{X}) - 1\bigr) = \Delta\mathbf{H}\bigl(\mathbb{X}, \beta(\mathbb{X}) - 1\bigr) = \Delta\mathbf{H}\bigl(\mathbb{X}, \beta(\mathbb{X})\bigr) + 1$$
$$= d_1 + 2m - 1 - \beta(\mathbb{X}). \quad \square$$

It follows from Claim 1 that $1 \leqslant d \leqslant m$. Notice that the difference function of $\mathbb{X}$ has the 'tail of a $\mathrm{CI}(d, \beta(\mathbb{X}))$'. Hence, by virtue of the proof of Theorem 12 in [29], we obtain that $\mathbb{X}_\ell := \mathbb{Y}$ is a $\mathrm{CI}(d, \beta(\mathbb{X}))$. Let $\mathcal{T}^\ell$ be the 2-type vector associated to the Hilbert function of $\mathbb{X}_\ell = \mathrm{CI}(d, \beta(\mathbb{X}))$. From Claim 2, we get

$$\sigma\bigl(\mathcal{T}^\ell\bigr) = d + \beta(\mathbb{X}) - 1 = \bigl(d_1 + 2m - 1 - \beta(\mathbb{X})\bigr) + \beta(\mathbb{X}) - 1 = d_1 + 2m - 2 = d_m.$$

Hence we have that $\mathcal{T}^\ell = (d_{m-d+1}, \dots, d_m)$. Furthermore, we see from Proposition 3.16 in [14] that $\mathbf{H}_{\mathbb{X}\setminus\mathbb{Y}} = \rho_2((d_1, \dots, d_{m-d}))$. So, using the induction hypothesis, it follows that $\mathbb{X} \setminus \mathbb{Y} = \bigcup_{i=1}^{\ell-1} \mathbb{X}_i$ is a skew ci configuration such that $(d_1, \dots, d_{m-d}) = \bigcup_{i=1}^{\ell-1} \mathcal{T}^i$, where $\mathcal{T}^i$ $(1 \leqslant i \leqslant \ell - 1)$ is the 2-type vector associated to the Hilbert function of a ci $\mathbb{X}_i$.

Let $\mathbb{H}_\ell$ be the hypersurface defined by D. Then, since $\mathbb{X}_\ell = \mathrm{CI}(d, \beta(\mathbb{X}))$, we have that $\alpha_{\mathbb{H}_\ell}(\mathbb{X}_\ell) = \beta(\mathbb{X}) = d_{m-d+1} + d - 1$. Hence, since

$$\begin{aligned}\sigma(\mathbb{X}_{\ell-1}) + \alpha(\mathbb{X}_\ell) &= \sigma\bigl(\mathcal{T}^{\ell-1}\bigr) + \alpha\bigl(\mathcal{T}^\ell\bigr) = d_{m-d} + d \\ &= (d_{m-d+1} - 2) + d < (d_{m-d+1} - 1) + d = \alpha_{\mathbb{H}_\ell}(\mathbb{X}_\ell),\end{aligned}$$

we obtain that $\mathbb{X} = \bigcup_{i=1}^{\ell} \mathbb{X}_i$ is a skew ci configuration and $\mathcal{T} = \bigcup_{i=1}^{\ell} \mathcal{T}^i$. $\square$

Example 4.8. Let $\mathcal{T} = (1, 3, 5)$ and let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^2$ with Hilbert function $\rho_2(\mathcal{T}) : 1\ 3\ 6\ 8\ 9\ 9\ \rightarrow$. All the decompositions of $\mathcal{T}$ are as follows:

$$(1, 3, 5), \quad (1, 3) \cup (5), \quad (1) \cup (3, 5) \quad \text{and} \quad (1) \cup (3) \cup (5).$$

Hence, by Theorem 4.7, we have that

$$\begin{aligned}&\mathbb{X} = \mathrm{CI}(3, 3), \quad \text{or} \quad \mathbb{X} = \mathrm{CI}(2, 2) \cup \mathrm{CI}(1, 5), \quad \text{or} \\ &\mathbb{X} = \mathrm{CI}(1, 1) \cup \mathrm{CI}(2, 4), \quad \text{or} \quad \mathbb{X} = \mathrm{CI}(1, 1) \cup \mathrm{CI}(1, 3) \cup \mathrm{CI}(1, 5).\end{aligned}$$

Example 4.9. Let $\mathcal{T} = (1, 3, 5, 8, 10) = (1, 3, 5) \cup (8, 10) = \mathcal{T}^1 \cup \mathcal{T}^2$ and let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^2$ with Hilbert function $\rho_2(\mathcal{T})$,

$$\rho_2(\mathcal{T}) = 1\ 3\ 6\ 10\ 15\ 19\ 22\ 24\ 26\ 27\ 27\ \rightarrow .$$

All the decompositions of $\mathcal{T}^2 = (8, 10)$ are as follows:

$$(8, 10) \quad \text{and} \quad (8) \cup (10)$$

Let $\mathbb{X}_2$ be a finite set of points in $\mathbb{P}^2$ with Hilbert function $\rho_2(\mathcal{T}^2)$. Then, by Theorem 4.7, we see that

$$\mathbb{X}_2 = \mathrm{CI}(2, 9) \quad \text{or} \quad \mathbb{X}_2 = \mathrm{CI}(1, 8) \cup \mathrm{CI}(1, 10).$$

Thus, using Theorem 4.7 again and Example 4.8, we have that $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2$:

$$\begin{aligned}
&1) \quad \mathbb{X} = \mathrm{CI}(3, 3) \cup \mathrm{CI}(2, 9), \text{ or} \\
&2) \quad \mathrm{CI}(2, 2) \cup \mathrm{CI}(1, 5) \cup \mathrm{CI}(2, 9), \text{ or} \\
&3) \quad \mathrm{CI}(1, 1) \cup \mathrm{CI}(2, 4) \cup \mathrm{CI}(2, 9), \text{ or} \\
&4) \quad \mathrm{CI}(1, 1) \cup \mathrm{CI}(1, 3) \cup \mathrm{CI}(1, 5) \cup \mathrm{CI}(2, 9), \text{ or} \\
&5) \quad \mathrm{CI}(3, 3) \cup \mathrm{CI}(1, 8) \cup \mathrm{CI}(1, 10), \text{ or} \\
&6) \quad \mathrm{CI}(2, 2) \cup \mathrm{CI}(1, 5) \cup \mathrm{CI}(1, 8) \cup \mathrm{CI}(1, 10), \text{ or} \\
&7) \quad \mathrm{CI}(1, 1) \cup \mathrm{CI}(2, 4) \cup \mathrm{CI}(1, 8) \cup \mathrm{CI}(1, 10), \text{ or} \\
&8) \quad \mathrm{CI}(1, 1) \cup \mathrm{CI}(1, 3) \cup \mathrm{CI}(1, 5) \cup \mathrm{CI}(1, 8) \cup \mathrm{CI}(1, 10).
\end{aligned}$$

So, any set of 27 points in $\mathbb{P}^2$ with Hilbert function

$$1\ 3\ 6\ 10\ 15\ 19\ 22\ 24\ 26\ 27\ 27\ \rightarrow$$

has to be of one of the 8 types above.

Notice also that for each type, Corollary 2.19 also gives us the resolution.

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Further reading

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